Nonlinear Size-Structured Population Models

A Dissertation
Presented to the
Graduate Faculty of the
University of Louisiana at Lafayette
In Partial Fulfillment of the
Requirements for the Degree
Doctor of Philosophy

Xubo Wang
Summer 2005

On Some Size-Structured Population Models

Xubo Wang

APPROVED:

Dr. Azmy S. Ackleh, Chairman
Professor of Mathematics

Dr. Keng Deng, Chairman
Professor of Mathematics

Dr. Sophia Jang
Assistant Professor of Mathematics

Dr. Robert D. Sidman
Professor of Mathematics

Dr. C. E. Palmer
Dean of the Graduate School

ACKNOWLEDGMENTS

I would like to express my sincere appreciation to my joint advisors Dr. Azmy S. Ackleh and Dr. Keng Deng for introducing me to the field of mathematical biology. They have shown me how exciting and fruitful interdisciplinary collaborations can be. Their perseverance, demand for continual improvement, and precise attitude towards research have impressed me strongly during my study. I would like to thank them for their continual guidance and encouragement. Their invaluable advice and continual encouragement have been crucial to the final version of this work.

I would like to make an acknowledgment to Dr. Sophia Jang and Dr. Robert D. Sidman who have served on my dissertation committee and have taught me several courses. I thank them for continually supporting me and generously helping me in many ways.

I thank all of my professors, both past and present. I have learned much more from them than they will ever realize. I also thank all of my friends who have made my life more enjoyable and more memorable.

I am extremely grateful to my great parents, Tingyan Wang and Xiue Zhang, for their faithful encouragement and support throughout every step of my life. Their influence can be seen on every day of my life. I have greatly profited from the deep concern and thoughtful suggestions from my dear brother Xutao Wang. Without the love, support, encouragement and understanding from my parents, my sisters and my brother, this work certainly would not have been possible.

I dedicate this dissertation to my beloved family for their deep love.

Contents

Preface

It is known that individuals in a biological population differ with regard to their physiological and behavioral characteristics, and, therefore, in the way they interact with their environment. As a result, vital processes such as birth, death, growth, metabolism, resource consumption, and so on, vary among individuals. The vital rates of individuals ultimately determine the dynamics of the entire population and how those dynamics are affected by the physical and biological environment. Therefore, accurate models of population level dynamics require a connection to individual level vital rates. One such connection is provided by so-called "size-structured" population models.

Theoretical studies of these models are valuable because such models help us to understand the dynamics of biological populations and because such models may yield insight into other dynamical behaviors.

Size-structured population models have been widely investigated in recent years. However, lots of problems still remain open, which motivates us to study some of them.

A quasilinear size-structured model that describes the dynamics of a population with n competing ecotypes is studied in Chapter 1. Under the assumption that the vital rates of each subpopulation depend on the total population due to competition, the conditions on the individual rates that guarantee competitive exclusion in the case of closed reproduction are provided. In particular, the results suggest that the ratio of the reproduction and mortality rates is a good measure to determine the winning ecotype. Meanwhile, in the case of open reproduction the coexistence of all ecotypes is established.

A model that describes the dynamics of the phytoplankton and zooplankton prey-predator system within the context of phytoplankton aggregation is considered in Chapter 2. Existence-uniqueness results of the solution are established via a comparison principle and the upper-lower solution technique.

A size-structured model that describes the dynamics of n-subpopulation with nonlinear growth, reproduction and mortality rates is investigated in Chapter 3. Existence and uniqueness results for the solutions were established. The existence of a compact global attractor for the trajectories of the dynamical system defined by the solutions of this model was obtained. In addition, two special cases under open reproduction were considered and the asymptotic dynamics for these special cases were discussed.

The dissertation is partially supported by the National Science Foundation under grants # DMS-0311969 and # DMS-0211412 for which we are thankful.

Chapter 1

Competitive Exclusion and Coexistence for a Quasilinear Size-Structured Population Model

In this chapter, we present a quasilinear size-structured model that describes the dynamics of a population with n competing ecotypes. We assume that the vital rates of each subpopulation depend on the total population due to competition. We provide conditions on the individual rates that guarantee competitive exclusion in the case of closed reproduction (offspring always belongs to the same ecotype as the parent). In particular, our results suggest that the ratio of the reproduction and mortality rates is a good measure to determine the winning ecotype. Meanwhile, we show that in the case of open reproduction, all ecotypes coexist.

1.1 Introduction

The Competitive Exclusion Principle asserts that no two populations competing for a common resource can live indefinitely in the same ecological niche. The validity of this principle has been investigated for many mathematical models that include both structured and non-structured populations (e.g., [2, 3, 4, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24]).

The results of this chapter have been published by Mathematical Biosciences [**192** (2004), 177-192].

On the side of non-structured population models, the literature contains many results. Below we briefly discuss a few of them. In [4, 20, 24], the authors investigate a competitive Lotka-Volterra system of equations and provide conditions on the parameters that guarantee that all but one of the species are driven to extinction. In [2], a generalized logistic model was developed. This model is composed of a continuum of subpopulations each with its own growth and mortality rates. Using the theory of weak convergence of probability measures, the authors show that the Competitive Exclusion Principle is valid for their model. In [3], a predator-prey Lotka-Volterra model which consists of many predator-prey subpopulations was studied. Therein, the authors show that all subpopulations become extinct except for the predator-prey pair which optimizes the growth to mortality ratio. In [19], the author studies the global stability of a boundary equilibrium (which corresponds to the extinction of one competing species) for a general three-dimensional competition model with two competing predators. The global stability is achieved by the construction of an appropriate Lyapunov function, which is a modification of those introduced by others.

In [10], the authors consider an n-pathogen, single host model. They show that pathogen strains with differing levels of virulence die out asymptotically except for those that optimize the basic reproduction number. In [1], the authors study an n-pathogen, single host model with variable population size. They prove that if the model parameters satisfy certain inequalities, then competition between n pathogens for a single host leads to the exclusion of all pathogens except the one with the largest basic reproduction number. In addition, they give an example that shows that if these inequalities are not satisfied, then coexistence may occur. In [13], the authors study a two-sex, susceptible-infective-susceptible sexually transmitted disease model with two competing strains. Therein, they investigate the existence and stability of the boundary equilibria that characterize the competitive exclusion of these two strains; they also investigate the existence and stability of the positive coexistence equilibrium, which characterizes the possibility of coexistence of the two strains. They obtain sufficient and necessary conditions for the existence and global stability of these equilibria.

For structured populations, considerably less work has been done due to the complexity of these models. In [23], competitive exclusion is proved for a discrete-time, size-structured, nonlinear matrix model of m competing species in a chemostat. The winner is the population that is able to grow at the lowest nutrient concentration. In [17], age and age-size structured population models composed of n ecotypes were studied. The authors show that a good measure of "ecotype fitness" is the product of the reproduction and survivorship functions.

This chapter is organized as follows. In Section 1.2, we present the population model. In Section 1.3, we establish existence and uniqueness results for the model. In Section 1.4, under the closed reproduction we provide conditions on the individuals rates which guarantee competitive exclusion, while under the open reproduction we show that all ecotypes survive. Further discussion and some numerical results are given in Section 1.5.

1.2 The population model

We consider a species with n competing ecotypes. For $i = 1, 2, \ldots, n$, we describe the dynamics of the subpopulation consisting of individuals of the i^{th} ecotype with the following individual size-structured model of the McKendrick-von Foerster type

$$\begin{aligned}
&(u_i)_t + g_i(P(t))(u_i)_x + m_i(P(t))u_i = 0 && 0 < x < \infty, \quad t > 0, \\
&g_i(P(t))u_i(0,t) = \sum_{j=1}^{n} \int_0^\infty \gamma_{i,j}\beta_j(P(t))u_j(x,t)dx && t > 0, \\
&u_i(x,0) = u_{i0}(x) && 0 \le x < \infty.
\end{aligned} \tag{1.2.1}$$

Here $u_i(x,t)$, $i = 1, 2, \ldots n$, is the density of individuals of the i^{th} ecotype having size x at time t, and $P(t) = \displaystyle\sum_{i=1}^{n} \int_0^\infty u_i(x,t)dx$ is the total number of individuals in the population at time t. The functions g_i, m_i, and β_i denote respectively the growth rate, the mortality rate, and the reproduction rate of an individual in the i^{th} subpopulation. These individual rates depend on the total number of individuals in the population. The constant parameter $0 \leq \gamma_{i,j} \leq 1$ represents the probability that an individual of the j^{th} ecotype will reproduce an individual of the i^{th} ecotype. Clearly, $\sum_{j=1}^{n} \gamma_{i,j} = \sum_{i=1}^{n} \gamma_{i,j} = 1, 1 \leq i, j \leq n$. In this chapter, we focus on the asymptotic behavior of the population in two cases. The first case is that all ecotypes are closed under reproduction in which offspring always belongs to the same ecotype as the parent, i.e., $\gamma_{i,i} = 1$ and $\gamma_{i,j} = 0$ for $i \neq j$. The second case is that ecotypes are open under reproduction in which individuals of ecotype i may reproduce individuals of ecotype j.

Linear models of type (1.2.1) have been used to describe the dynamics of mosquitofish populations in rice fields [6]. Simulation studies therein demonstrate that solutions to such models could lead to population densities that exhibit dispersion and bimodality as field data suggested in [9]. Such dispersion and bimodality cannot result from the classical size-structured model (i.e., all individuals are assumed to be of the same ecotype) except under some biologically unrealistic conditions (see [7]). This indicates that the consideration of several ecotypes is important if such size-structured models are to be used as prediction tools.

Rigorous theoretical developments of inverse problems associated with linear models of type (1.2.1) were given in [7, 15, 16]. In [8], such inverse methodology was used for estimating the distribution of individual growth rates based on aggregate population data. Therein, a good fit of the model to field data was presented. A survey of results and other references for such models can be found in [5].

1.3 Existence and uniqueness results

Throughout the discussion, we assume that the parameters in (1.2.1) satisfy the following:

(H1) $g_i(P)$ is strictly positive and continuously differentiable for $0 < P < \infty$.

(H2) $m_i(P)$ is nonnegative and continuously differentiable for $0 \leq P < \infty$.

(H3) $\beta_i(P)$ is continuously differentiable and uniformly bounded for $0 \leq P < \infty$ with

$0 \le \beta_i \le \beta_M$.

(H4) $u_{i0} \in L^1(0,\infty)$ and $u_{i0} \ge 0$.

In the spirit of [6], we use the contraction mapping argument to discuss the existence-uniqueness of solutions to problem (1.2.1). We begin with the definition of the solution.

Definition 1.3.1. A nonnegative function $u(x,t) = (u_1(x,t), u_2(x,t), \dots, u_n(x,t))$ on $[0,\infty) \times [0,T)$, with $u(\cdot,t)$ integrable is a solution of (1.2.1) if $P(t) := \sum_{i=1}^{n} \int_0^\infty u_i(x,t)dx$ is a continuous function on $[0,T)$ and for $i = 1,2,\dots,n$, $u_i(x,t)$ satisfies $(1.2.1)_2$, $(1.2.1)_3$, and the equation

$$Du_i(x,t) = -m_i(P(t))u_i(x,t) \qquad 0 < x < \infty,\ 0 < t < T \tag{1.3.1}$$

with

$$Du_i(x,t) = \lim_{h\to 0} \frac{u_i(X_i(t+h;x,t),t+h) - u_i(x,t)}{h}, \tag{1.3.2}$$

where $X_i(t;x_0,t_0)$ is the solution of the equation for the characteristic curves given by

$$\begin{cases} \dfrac{d}{dt}x(t) = g_i(P(t)) \\ x(t_0) = x_0. \end{cases} \tag{1.3.3}$$

From (H1) it follows that the function X_i is strictly increasing. Hence, a unique inverse function $\tau_i(x;x_0,t_0)$ exists. Let $z_i(t) = X_i(t;0,0)$ where $(z_i(t),t)$ represents the characteristic curve passing through $(0,0)$ and dividing the (x,t)-plane into two parts.

Let $B_i(t) := \sum_{j=1}^{n} \int_0^\infty \gamma_{i,j}\beta_j(P(t))u_j(x,t)dx$, the inflow of newborns in the i^{th} subpopulation at time t. Using the method of characteristics, we reduce problem (1.2.1) to a system of coupled equations for $P(t)$ and $B_i(t)$.

Integrating (1.3.1) along the characteristics, we have

$$\begin{aligned} u_i(x,t) &= \frac{B_i(\tau_i(0;x,t))}{g_i(P(\tau_i(0;x,t)))} \exp\left(-\int_{\tau_i(0;x,t)}^{t} m_i(P(s))ds\right) && x < z_i(t), \\ u_i(x,t) &= u_{i0}(X_i(0;x,t)) \exp\left(-\int_0^t m_i(P(s))ds\right) && x \ge z_i(t). \end{aligned} \tag{1.3.4}$$

Then integrating (1.3.4) with respect to x and summing over the indices $i = 1, 2, \ldots, n$, we obtain an integral equation for $P(t)$,

$$\begin{aligned} P(t) &= \sum_{i=1}^{n} \left[\int_0^{z_i(t)} \frac{B_i(\tau_i(0;x,t))}{g_i(P(\tau_i(0;x,t)))} \exp\left(- \int_{\tau_i(0;x,t)}^{t} m_i(P(s))ds \right) dx \right. \\ &\quad \left. + \int_{z_i(t)}^{\infty} u_{i0}(X_i(0;x,t)) \exp\left(- \int_0^t m_i(P(s))ds \right) dx \right] \\ &= \sum_{i=1}^{n} \left[\int_0^t B_i(\eta) e^{-\int_\eta^t m_i(P(s))ds} d\eta + \int_0^\infty u_{i0}(\xi) e^{-\int_0^t m_i(P(s))ds} d\xi \right]. \end{aligned} \tag{1.3.5}$$

Similarly, substituting (1.3.4) in the definition of $B_i(t)$, we obtain an integral equation for $B_i(t)$, $i = 1, 2, \ldots, n$,

$$\begin{aligned} B_i(t) = \sum_{j=1}^{n} &\left[\int_0^t \gamma_{i,j} \beta_j(P(t)) B_j(\eta) e^{-\int_\eta^t m_j(P(s))ds} d\eta \right. \\ &\left. + \int_0^\infty \gamma_{i,j} \beta_j(P(t)) u_{j0}(\xi) e^{-\int_0^t m_j(P(s))ds} d\xi \right]. \end{aligned} \tag{1.3.6}$$

Clearly, if $P(t)$ and $B_i(t)$ are nonnegative continuous solutions of (1.3.5)-(1.3.6), then $u(x,t)$ defined by (1.3.4) is a solution of (1.2.1). Since we have established a correspondence between (1.2.1) and (1.3.5)-(1.3.6), to obtain the existence and uniqueness results for problem (1.2.1), we only need to study the solvability of the system of integral equations (1.3.5)-(1.3.6). To this end, for $K > ||u_0||_{L^1} = \sum_{i=1}^{n} \int_0^\infty u_{i0}(x)dx$, let $S_{T,K} = \{f(t) \in C[0,T] | f(0) = ||u_0||_{L^1},\ 0 \le f(t) \le K\}$. For each $P \in S_{T,K}$, let $B_i(t) \in C[0,T]$ be the unique nonnegative solution of the linear Volterra integral equation (1.3.6), and we define the operator $\mathcal{P} : S_{T,K} \to C[0,T]$ in such a way that $\mathcal{P}(P)(t)$ is the right hand side of (1.3.5) for these $P(t)$ and $B_i(t)$.

Lemma 1.3.2. Suppose that hypotheses (H1)-(H4) hold. Then there exists a value $T > 0$ for which $\mathcal{P}$ *has a unique fixed point.*

Proof. We first show that $\mathcal{P}$ maps $S_{T,K}$ into itself. To this end, we obtain a function to bound $B_i(t)$. By (1.3.6) and the hypotheses (H2), (H3), we have

$$B_i(t) \le \beta_M \sum_{j=1}^{n} \int_0^t B_j(\eta)d\eta + \beta_M \|u_0\|_{L^1}.$$

Thus,

$$\sum_{j=1}^{n} B_j(t) \le n\beta_M \int_0^t \sum_{j=1}^{n} B_j(\eta) d\eta + n\beta_M \|u_0\|_{L^1},$$

which by Gronwall's inequality implies

$$\sum_{j=1}^{n} B_j(t) \le n\beta_M \|u_0\|_{L^1} e^{n\beta_M t}. \tag{1.3.7}$$

A combination of (1.3.5) and (1.3.7) then yields

$$\begin{aligned}\mathcal{P}(P)(t) &\le \int_0^t \sum_{j=1}^{n} B_j(\eta) d\eta + ||u_0||_{L^1} \\ &\le n\beta_M \|u_0\|_{L^1} \int_0^t e^{n\beta_M \eta} d\eta + ||u_0||_{L^1} \\ &\le e^{n\beta_M T} ||u_0||_{L^1} \le K,\end{aligned}$$

provided T is very small.

We next show that $\mathcal{P}$ is contractive. For any $P, \hat{P} \in S_{T,K}$, letting B_i and $\hat{B}_i$ be the solutions of (1.3.6) for P and $\hat{P}$, respectively, we have

$$\begin{aligned}&|\mathcal{P}(P)(t) - \mathcal{P}(\hat{P})(t)| \\ &\quad = \Bigg| \sum_{j=1}^{n} \int_0^t B_j(\eta) e^{-\int_\eta^t m_j(P(s))ds}\, d\eta - \sum_{j=1}^{n} \int_0^t \hat{B}_j(\eta) e^{-\int_\eta^t m_j(\hat{P}(s))ds} d\eta \\ &\qquad + \sum_{j=1}^{n} \int_0^\infty u_{j0}(\xi) \left[e^{-\int_0^t m_j(P(s))ds} - e^{-\int_0^t m_j(\hat{P}(s))ds} \right] d\xi \Bigg| \\ &\quad \le \sum_{j=1}^{n} \int_0^t \left| B_j(\eta) - \hat{B}_j(\eta) \right| d\eta \\ &\qquad + \sum_{j=1}^{n} \int_0^t \hat{B}_j(\eta) \int_\eta^t \left| m_j(P(s)) - m_j(\hat{P}(s)) \right| ds d\eta \\ &\qquad + \sum_{j=1}^{n} \int_0^\infty u_{j0}(\xi) \int_0^t \left| m_j(P(s)) - m_j(\hat{P}(s)) \right| ds d\xi.\end{aligned} \tag{1.3.8}$$

We now estimate each integral in the last expression of (1.3.8). Let $|F_i(t)| = |B_i(t) - \hat{B}_i(t)|$. Then from (1.3.6) and (1.3.7), we have

$$
\begin{aligned}
|F_i(t)| \leq \sum_{j=1}^{n} \Bigg| & \int_0^t \gamma_{i,j}\beta_j(P(t))B_j(\eta)e^{-\int_\eta^t m_j(P(s))ds}d\eta - \int_0^t \gamma_{i,j}\beta_j(P(t))\hat{B}_j(\eta)e^{-\int_\eta^t m_j(P(s))ds}d\eta \\
& + \int_0^t \gamma_{i,j}\beta_j(P(t))\hat{B}_j(\eta)e^{-\int_\eta^t m_j(P(s))ds}d\eta - \int_0^t \gamma_{i,j}\beta_j(P(t))\hat{B}_j(\eta)e^{-\int_\eta^t m_j(\hat{P}(s))ds}d\eta \\
& + \int_0^t \gamma_{i,j}\beta_j(P(t))\hat{B}_j(\eta)e^{-\int_\eta^t m_j(\hat{P}(s))ds}d\eta - \int_0^t \gamma_{i,j}\beta_j(\hat{P}(t))\hat{B}_j(\eta)e^{-\int_\eta^t m_j(\hat{P}(s))ds}d\eta \Bigg| \\
& + \sum_{j=1}^{n} \left[\int_0^\infty \gamma_{i,j} \left| \beta_j(P(t))e^{-\int_0^t m_j(P(s))ds} - \beta_j(\hat{P}(t))e^{-\int_0^t m_j(\hat{P}(s))ds} \right| u_{j0}(\xi)d\xi \right] \\
\leq & \sum_{j=1}^{n} \beta_M \int_0^t \left| B_j(\eta) - \hat{B}_j(\eta) \right| d\eta + \sum_{j=1}^{n} \beta_M \int_0^t \hat{B}_j(\eta) \int_\eta^t \left| m_j(P(s)) - m_j(\hat{P}(s)) \right| ds d\eta \\
& + \sum_{j=1}^{n} \int_0^t \hat{B}_j(\eta)|\beta_j(P(t)) - \beta_j(\hat{P}(t))|d\eta \\
& + \sum_{j=1}^{n} \int_0^\infty \left| \beta_j(P(t))e^{-\int_0^t m_j(P(s))ds} - \beta_j(\hat{P}(t))e^{-\int_0^t m_j(\hat{P}(s))ds} \right| u_{j0}(\xi)d\xi,
\end{aligned}
$$

or equivalently,

$$
|F_i(t)| \leq \sum_{j=1}^{n} \beta_M \int_0^t |F_j(\eta)|d\eta + G_i(t). \tag{1.3.9}
$$

Here

$$
\begin{aligned}
G_i(t) = & \sum_{j=1}^{n} \beta_M \int_0^t \hat{B}_j(\eta) \int_\eta^t \left| m_j(P(s)) - m_j(\hat{P}(s)) \right| ds d\eta \\
& + \sum_{j=1}^{n} \int_0^t \hat{B}_j(\eta)|\beta_j(P(t)) - \beta_j(\hat{P}(t))|d\eta \\
& + \sum_{j=1}^{n} \int_0^\infty \left| \beta_j(P(t))e^{-\int_0^t m_j(P(s))ds} - \beta_j(\hat{P}(t))e^{-\int_0^t m_j(\hat{P}(s))ds} \right| u_{j0}(\xi)d\xi \\
\leq & \sum_{j=1}^{n} \beta_M \int_0^t \hat{B}_j(\eta) \int_\eta^t \left| m_j(P(s)) - m_j(\hat{P}(s)) \right| ds d\eta \\
& + \beta_K |P(t) - \hat{P}(t)| \sum_{j=1}^{n} \int_0^t \hat{B}_j(\eta)d\eta \\
& + \sum_{j=1}^{n} \int_0^\infty \Bigg[\beta_M \int_0^t \left| m_j(P(s)) - m_j(\hat{P}(s)) \right| ds \\
& + \left| \beta_j(P(t)) - \beta_j(\hat{P}(t)) \right| \Bigg] u_{j0}(\xi)d\xi
\end{aligned}
$$

$$\leq (\beta_M m_K T + \beta_K)||P - \hat{P}||_\infty \sum_{j=1}^{n} \int_0^t \hat{B}_j(\eta) d\eta$$

$$+(\beta_M m_K T + \beta_K)||P - \hat{P}||_\infty ||u_0||_{L^1}$$

$$\leq (\beta_M m_K T + \beta_K)\, e^{n\beta_M T} \|u_0\|_{L^1} ||P - \hat{P}||_\infty$$

$$:= J(T)\|P - \hat{P}\|_\infty,$$

where $\beta_K = \sup_{P\in[0,K], 1\leq i\leq n} |\beta_i'(P)|$ and $m_K = \sup_{P\in[0,K], 1\leq i\leq n} |m_i'(P)|$.

Thus, from (1.3.9) we obtain

$$|F_i(t)| \leq \sum_{j=1}^{n} \beta_M \int_0^t |F_j(\eta)| d\eta + J(T)\|P - \hat{P}\|_\infty.$$

Summing the above inequality over the indices $i = 1, 2, \ldots, n$, we find

$$\sum_{i=1}^{n} |F_i(t)| \leq n\beta_M \int_0^t \sum_{j=1}^{n} |F_j(\eta)| d\eta + nJ(T)\|P - \hat{P}\|_\infty,$$

which by Gronwall's inequality leads to

$$\sum_{i=1}^{n} |F_i(t)| \leq nJ(T) e^{n\beta_M T} \|P - \hat{P}\|_\infty.$$

Hence, we have

$$\sum_{j=1}^{n} \int_0^t \left| B_j(\eta) - \hat{B}_j(\eta) \right| d\eta \leq nJ(T) e^{n\beta_M T} T \|P - \hat{P}\|_\infty.$$

On the other hand, we find that

$$\sum_{j=1}^{n} \int_0^t \hat{B}_j(\eta) \int_\eta^t \left| m_j(P(s)) - m_j(\hat{P}(s)) \right| ds d\eta$$

$$\leq m_K T \|P - \hat{P}\|_\infty \int_0^t \sum_{j=1}^{n} \hat{B}_j(\eta) d\eta$$

$$\leq m_K T \|u_0\|_{L^1} e^{n\beta_M T} \|P - \hat{P}\|_\infty$$

and

$$\sum_{j=1}^{n} \int_0^\infty u_{j0}(\xi) \int_0^t \left| m_j(P(s)) - m_j(\hat{P}(s)) \right| ds d\xi$$

$$\leq m_K T ||u_0||_{L^1} \|P - \hat{P}\|_\infty.$$

Therefore, $\mathcal{P}$ is contractive provided that T is sufficiently small. The proof is thus completed. □

From the unique existence of the solution $P(t)$ and $B_i(t)$ of system (1.3.5)-(1.3.6), it follows that the solution of problem (1.2.1) must be unique because each $u_i(x,t)$ given by (1.3.4) is uniquely determined by $P(t)$ and $B_i(t)$. Thus, we have the following local existence result.

Theorem 1.3.3. Suppose that hypotheses (H1)-(H4) hold. Then there exists a value $T > 0$ such that problem (1.2.1) has a unique solution up to time T.

In order to establish the global existence result for problem (1.2.1), we introduce an upper bound on $P(t)$ for $t \in [0,T]$.

Lemma 1.3.4. Let $u(x,t)$ be a solution of (1.2.1) up to time T. Then $P(t)$ satisfies the following bound

$$P(t) \le \|u_0\|_{L^1} e^{\beta_M t} \quad \text{for } t \in [0,T]. \tag{1.3.10}$$

Proof. Let $P_i(t) = \displaystyle\int_0^\infty u_i(x,t)dx$ and $P(t) = \displaystyle\sum_{i=1}^n P_i(t)$. Integrating (1.3.4) with respect to x, we obtain an integral equation for $P_i(t)$, $i = 1, 2, \ldots, n$,

$$P_i(t) = \int_0^t B_i(\eta) e^{-\int_\eta^t m_i(P(s))ds} d\eta + \int_0^\infty u_{i0}(\xi) e^{-\int_0^t m_i(P(s))ds} d\xi. \tag{1.3.11}$$

Then differentiating (1.3.11) with respect to t, we have

$$P_i'(t) = \sum_{j=1}^n \gamma_{i,j}\beta_j(P)P_j - m_i(P)P_i. \tag{1.3.12}$$

Thus,

$$\begin{aligned} P'(t) &= \sum_{i=1}^n \left(\sum_{j=1}^n \gamma_{j,i}\beta_i(P) - m_i(P) \right) P_i \\ &= \sum_{i=1}^n (\beta_i(P) - m_i(P))P_i \\ &\le \sum_{i=1}^n \beta_i(P)P_i \\ &\le \beta_M P(t). \end{aligned}$$

Integrating the above relation over $(0,t)$ yields (1.3.10). □

Theorem 1.3.5. Suppose that hypotheses (H1)-(H4) hold; then problem (1.2.1) has a unique solution for all positive time.

The proof is essentially the same as that of Theorem 3 in [6] and hence is omitted.

1.4 Asymptotic behavior

Throughout this section, we assume an additional condition on the reproduction and the mortality rates.

(H5) $\beta_i(P)$ is nonincreasing and $m_i(P)$ is increasing for $0 \le P < \infty$, and there exists P_i^* such that $\beta_i(P_i^*) = m_i(P_i^*),\ i = 1, 2, \dots, n$.

In order to study the asymptotic behavior of the population, we consider the following system of coupled ordinary differential equations:

$$P_i'(t) = \sum_{j=1}^{n} \gamma_{i,j}\beta_j(P)P_j - m_i(P)P_i, \quad P_i(0) > 0, \quad i = 1, 2, \dots, n. \tag{1.4.1}$$

We first show that the total population $P(t)$ is uniformly bounded.

Lemma 1.4.1. Let $\overline{P} = \max_{1\le i\le n} P_i^*$ and $\underline{P} = \min_{1\le i\le n} P_i^*$. For any $0 < \varepsilon < 1$, define $I_\varepsilon = [\underline{P}(1-\varepsilon), \overline{P}(1+\varepsilon)]$. Then there exists a finite time t_ε^* such that $P \in I_\varepsilon$ for $t \ge t_\varepsilon^*$.

Proof. Summing (1.4.1) over the indices $i = 1, 2, \dots, n$, we have

$$P'(t) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \gamma_{j,i}\beta_i(P) - m_i(P) \right) P_i = \sum_{i=1}^{n} \left(\beta_i(P) - m_i(P)\right) P_i. \tag{1.4.2}$$

If $P > \overline{P}(1+\varepsilon)$, then $\beta_i(P) - m_i(P) \le -\tilde{\theta}_\varepsilon$ with $\tilde{\theta}_\varepsilon > 0$ for $i = 1, 2, \dots, n$. By (1.4.2), $P' \le -\tilde{\theta}_\varepsilon P$, i.e., P is strictly decreasing in t. Hence, there exists a value $\tilde{t}_\varepsilon$ such that $P \le \overline{P}(1+\varepsilon)$ for $t \ge \tilde{t}_\varepsilon$.

On the other hand, if $P < \underline{P}(1-\varepsilon)$, then $\beta_i(P) - m_i(P) \ge \hat{\theta}_\varepsilon$ with $\hat{\theta}_\varepsilon > 0$ for $i = 1, 2, \dots, n$. By (1.4.2), $P' \ge \hat{\theta}_\varepsilon P$, i.e., P is strictly increasing in t. Hence, there exists a value $\hat{t}_\varepsilon$ such that $P \ge \underline{P}(1-\varepsilon)$ for $t \ge \hat{t}_\varepsilon$.

Let $t_\varepsilon^* = \max\{\tilde{t}_\varepsilon, \hat{t}_\varepsilon\}$. Then $\underline{P}(1-\varepsilon) \le P(t) \le \overline{P}(1+\varepsilon)$ for all $t \ge t_\varepsilon^*$. □

1.4.1 Closed reproduction case

In this case, we assume that reproduction is closed under subpopulations, that is, individuals in the i^{th} subpopulation only reproduce individuals in the i^{th} subpopulation. Hence $\gamma_{i,i} = 1,\ \gamma_{i,j} = 0,\ i \neq j,\ i, j = 1, 2, \ldots, n$, and (1.4.1) takes the form:

$$P_i' = (\beta_i(P) - m_i(P))P_i, \quad P_i(0) > 0, \quad i = 1, 2, \ldots, n. \tag{1.4.3}$$

We now introduce a condition on the ratio of the reproduction and mortality rates.

(H6) $\dfrac{\beta_1(P)}{m_1(P)} > \dfrac{\beta_i(P)}{m_i(P)},\ i = 2, \ldots, n$, for any $P \in I_0 = [\underline{P}, \overline{P}]$.

The next result shows that under (H6) the subpopulations $P_2, \ldots, P_n$ become extinct as $t \to \infty$.

Lemma 1.4.2. Suppose that hypothesis (H6) holds. Then the solution of (1.4.3) satisfies that for each $i = 2, \ldots, n,\ P_i(t) \to 0$ as $t \to \infty$.

Proof. We first note that by the continuity of β_i and m_i, there exists an $\bar{\varepsilon}$ $(0 < \bar{\varepsilon} < 1)$ such that (H6) holds for $P \in I_{\bar{\varepsilon}}$. In particular, (H6) holds for $P = P_1^*$, which implies $\overline{P} = P_1^* > P_i^*,\ i = 2, \ldots, n$. To show that for $i = 2, \ldots, n,\ P_i(t) \to 0$ as $t \to \infty$, it suffices to show that for $i = 2, \ldots, n,\ \dfrac{P_i^{\sigma_i}}{P_1} \to 0$ as $t \to \infty$ for some positive constant σ_i. To this end, choose $0 < \varepsilon < \bar{\varepsilon}$ sufficiently small so that for any $P \in I_\varepsilon \subset I_{\bar{\varepsilon}}$ we have

$$(\sigma_i m_i(P) - m_1(P)) \left(\frac{\beta_1(P)}{m_1(P)} - 1 \right) \leq \frac{1}{2} \sigma_i m_i(P) \left(\frac{\beta_1(P)}{m_1(P)} - \frac{\beta_i(P)}{m_i(P)} \right),$$

where $\sigma_i = \min\limits_{P \in I_{\bar{\varepsilon}}} \dfrac{m_1(P)}{m_i(P)}$. Set $\Phi_i(t) = \dfrac{P_i^{\sigma_i}(t)}{P_1(t)}$. Φ_i satisfies that for $t \geq t_\varepsilon^*$,

$$\begin{aligned} \Phi_i' &= \frac{\sigma_i P_i^{\sigma_i - 1} P_i' P_1 - P_i^{\sigma_i} P_1'}{P_1^2} \\ &= \frac{\sigma_i P_i^{\sigma_i} \left(\beta_i(P) - m_i(P)\right) P_1 - P_i^{\sigma_i} \left(\beta_1(P) - m_1(P)\right) P_1}{P_1^2} \\ &= \left[\sigma_i \left(\beta_i(P) - m_i(P)\right) - \left(\beta_1(P) - m_1(P)\right)\right] \Phi_i \end{aligned} \tag{1.4.4}$$

$$
\begin{aligned}
&= \left[\sigma_i m_i(P)\left(\frac{\beta_i(P)}{m_i(P)} - 1\right) - m_1(P)\left(\frac{\beta_1(P)}{m_1(P)} - 1\right)\right]\Phi_i \\
&= \left[(\sigma_i m_i(P) - m_1(P))\left(\frac{\beta_1(P)}{m_1(P)} - 1\right) - \sigma_i m_i(P)\left(\frac{\beta_1(P)}{m_1(P)} - \frac{\beta_i(P)}{m_i(P)}\right)\right]\Phi_i \qquad (1.4.5)\\
&\le -\frac{1}{2}\sigma_i m_i(P)\left(\frac{\beta_1(P)}{m_1(P)} - \frac{\beta_i(P)}{m_i(P)}\right)\Phi_i \\
&\le -\lambda_i \Phi_i
\end{aligned}
$$

with positive λ_i. Integrating (1.4.5) from t_ε^* to t then yields

$$
\Phi_i(t) \le \Phi_i(t_\varepsilon^*)e^{-\lambda_i(t-t_\varepsilon^*)}, \qquad (1.4.6)
$$

where $\Phi_i(t_\varepsilon^*) = \dfrac{P_i^{\sigma_i}(t_\varepsilon^*)}{P_1(t_\varepsilon^*)} > 0$. Hence, $\Phi_i(t) \to 0$ as $t \to \infty$. □

We now show that P_1 converges to a positive equilibrium and hence is the winning ecotype. To this end, we fix ε as chosen in the proof of the above theorem.

Theorem 1.4.3. Suppose that hypothesis (H6) holds. Then $P_1(t) \to P_1^*$ as $t \to \infty$.

Proof. Consider the following initial value problem:

$$
\begin{cases}
y' = (\beta_1(y) - m_1(y))y, & t_\varepsilon^* < t < \infty, \\
y(t_\varepsilon^*) = P_1(t_\varepsilon^*).
\end{cases}
$$

Clearly, $y(t) \to P_1^*$ as $t \to \infty$. Furthermore, since $P_1' \le (\beta_1(P_1) - m_1(P_1))P_1$, by comparison $y(t) \ge P_1(t)$ for all $t \ge t_\varepsilon^*$. On the other hand, we have

$$
\begin{aligned}
\frac{d}{dt}\ln\left(\frac{P_1}{y}\right) &= \frac{P_1'}{P_1} - \frac{y'}{y} \\
&= (\beta_1(P) - m_1(P)) - (\beta_1(y) - m_1(y)) \\
&= (\beta_1(P_1) - m_1(P_1)) - (\beta_1(y) - m_1(y)) + (\beta_1'(\zeta) - m_1'(\zeta))\sum_{j=2}^{n} P_j \qquad (1.4.7)\\
&= \left(\beta_1'(\hat{\zeta}) - m_1'(\hat{\zeta})\right)(P_1 - y) + (\beta_1'(\zeta) - m_1'(\zeta))\sum_{j=2}^{n} P_j,
\end{aligned}
$$

where ζ is between P and P_1 and $\hat{\zeta}$ is between P_1 and y.

Since $\left(m_1'(\hat{\zeta}) - \beta_1'(\hat{\zeta})\right) \ge c > 0$ for $t \ge t_\varepsilon^*$, rewriting (1.4.7) we find

$$
y - P_1 \le \frac{1}{c}\left(\frac{d}{dt}\ln\left(\frac{P_1}{y}\right) + (m_1'(\zeta) - \beta_1'(\zeta))\sum_{j=2}^{n} P_j\right). \qquad (1.4.8)
$$

Integrating (1.4.8) from t_ε^* to t, we then obtain

$$\int_{t_\varepsilon^*}^t (y(\eta) - P_1(\eta))d\eta \leq \frac{1}{c}\left(\ln\left(\frac{P_1(t)}{y(t)}\right) + \sum_{j=2}^n \int_{t_\varepsilon^*}^t (m_1'(\zeta) - \beta_1'(\zeta))\, P_j(\eta)d\eta\right)$$
$$\leq M < \infty,$$

where M is independent of t, since $P_1(t)$ and $y_1(t)$ are both bounded by positive constants, and by (1.4.6) for $j = 2, \ldots, n$, $\int_{t_\varepsilon^*}^\infty P_j(t)dt < \infty$. This implies that $\int_{t_\varepsilon^*}^\infty (y(t) - P_1(t))dt < \infty$. Furthermore, it is easily seen that $(y(t) - P_1(t))'$ is bounded on $[t_\varepsilon^*, \infty)$. Hence, $y(t) - P_1(t) \to 0$ as $t \to \infty$, that is, $P_1(t) \to P_1^*$ as $t \to \infty$. □

1.4.2 Open reproduction case

In this case, we assume that reproduction is open under subpopulations, that is, individuals in the i^{th} subpopulation may also reproduce individuals in the j^{th} subpopulation. We will show that if the graph associated with the matrix $[\gamma_{i,j}]$ is strongly connected (the matrix $[\gamma_{i,j}]$ is irreducible), then all ecotypes of the population coexist. To this end, for the convenience of the reader, we assume the following:

(H7) $\gamma_{1,2} > 0,\ \gamma_{2,3} > 0,\ \ldots\ \gamma_{(n-1),n} > 0$, and $\gamma_{n,1} > 0$. Otherwise, $\gamma_{i,j} \geq 0$, $1 \leq i, j \leq n$.

The assumption (H7) implies that the matrix $[\gamma_{i,j}]$ is irreducible, and our following argument can be easily modified to apply in the case where (H7) is replaced by any other assumption that leads to a strongly connected graph associated with the matrix $[\gamma_{i,j}]$.

We now show that under (H7) all ecotypes coexist for all times.

Theorem 1.4.4. Suppose that hypothesis (H7) holds. Then there exists a positive constant c such that $\liminf_{t\to\infty} P_i(t) \geq c$ *for* $i = 1, 2, \ldots, n$.

Proof. To obtain the claimed lower bound, it suffices to show that there exists a $T_1^* > 0$ such that $P_i \geq c$ for $t \geq T_1^*$, $i = 1, 2, \ldots, n$. For simplicity, let $I = I_{1/2} = [\underline{P}/2, 3\overline{P}/2]$ and $T_0^* = t_{1/2}^*$. We first define $\Psi_1(t) = P_1(t)$. Taking (1.4.1) into account, we find that for $t \geq T_0^*$, Ψ_1 satisfies

$$\begin{aligned}\Psi_1' &= \sum_{j=1}^{n} \gamma_{1,j}\beta_j(P)P_j - m_1(P)P_1 \\ &\geq \gamma_{1,2}\beta_2(P)P_2 - m_1(P)P_1 \\ &\geq c_1(P_2 + \Psi_1) - d_1\Psi_1,\end{aligned} \tag{1.4.9}$$

where $c_1 = \min\left\{\gamma_{1,2}\min_{P\in I}\beta_2(P), \min_{P\in I} m_1(P)\right\}$ and $d_1 = 2\max_{P\in I} m_1(P)$. Then let $\Psi_2(t) = P_2 + \Psi_1$. Ψ_2 satisfies

$$\begin{aligned}\Psi_2' &= \sum_{j=1}^{n} \gamma_{2,j}\beta_j(P)P_j + \Psi_1' - m_2(P)P_2 \\ &\geq \gamma_{2,3}\beta_3(P)P_3 + c_1\Psi_2 - d_1\Psi_1 - m_2(P)P_2 \\ &\geq c_2(P_3 + \Psi_2) - d_2\Psi_2.\end{aligned} \tag{1.4.10}$$

Now let $\Psi_3(t) = P_3 + \Psi_2$. In a similar manner, we can see that Ψ_3 satisfies

$$\Psi_3' \geq c_3(P_4 + \Psi_3) - d_3\Psi_3. \tag{1.4.11}$$

Thus, we can define a sequence $\{\Psi_i(t)\}_{i=1}^n$ such that for $t \geq T_0^*$, $i = 1, 2, \ldots, n-1$,

$$\Psi_i' \geq c_i\Psi_{i+1} - d_i\Psi_i \tag{1.4.12}$$

and

$$\Psi_n = P_n + \Psi_{n-1} = P \geq \underline{P}, \tag{1.4.13}$$

where $c_i = \min\left\{c_{i-1}, \gamma_{i,i+1}\min_{P\in I}\beta_{i+1}(P)\right\}$ and $d_i = \max\left\{d_{i-1}, \max_{P\in I} m_i(P)\right\}$, $i = 2, \ldots, n-1$.

In view of (1.4.12) and (1.4.13), we have

$$\Psi_{n-1}' \geq \tilde{c}_{n-1} - d_{n-1}\Psi_{n-1}.$$

Integrating the above inequality from T_0^* to t, we find

$$\Psi_{n-1} \geq \frac{\tilde{c}_{n-1}}{d_{n-1}}\left(1 - e^{-d_{n-1}(t-T_0^*)}\right),$$

which implies that for $t \geq 2T_0^*$, $\Psi_{n-1} \geq \hat{c}_{n-1}$. Then by means of (1.4.12), we find that for $t \geq 3T_0^*$, $\Psi_{n-2} \geq \hat{c}_{n-2}$. Continuing in such a way, finally we obtain that for $t \geq nT_0^*$, $\Psi_1 \geq \hat{c}_1$, that is, $P_1 \geq \hat{c}_1$.

We then make use of (1.4.1) to find that for $t \geq nT_0^*$,

$$\begin{aligned} P_n' &\geq \gamma_{n,1}\beta_1(P)P_1 - m_n(P)P_n \\ &\geq \delta_n - d_nP_n, \end{aligned} \tag{1.4.14}$$

which, upon integration over (nT_0^*, t) yields that for $t \geq (n+1)T_0^*$, $P_n \geq \hat{\delta}_n$. Repeating this process in a backward manner, one can see that there exists a positive constant c such that $P_i \geq c$ for $t \geq T_1^* = 2nT_0^*$, $i = 1, 2, \ldots, n$. Thus, the proof is completed. □

1.5 Further discussion and numerical results

In the closed reproduction case, from Section 1.4.1 it is clear that subpopulations with smaller ratios $\beta_i(P)/m_i(P)$ will become extinct. This leads to the following question: What happens if two subpopulations have the same largest ratio? We will show that in this case, both subpopulations should survive. To this end, since all other subpopulations having smaller ratios will become extinct, we will focus on the following subsystem consisting of two subpopulations with the largest ratios $\beta_1(P)/m_1(P) = \beta_2(P)/m_2(P)$:

$$P_i' = (\beta_i(P) - m_i(P))P_i, \quad P_i(0) > 0, \quad i = 1, 2. \tag{1.5.1}$$

Because the ratios are equal, $P_1^* = P_2^*$ $(= P^*)$. If $P_1(0) + P_2(0) = P(0) < P^*$, then $P_i' > 0$, which means that both P_1 and P_2 will increase in t, and hence for $t \geq 0$, $P_i(0) \leq P_i(t) \leq P^*$, $i = 1, 2$. If $P_1(0) + P_2(0) = P(0) = P^*$, then $P_1 = P_1(0)$ and $P_2 = P_2(0)$ for $t \geq 0$. Finally, if $P_1(0) + P_2(0) = P(0) > P^*$, it is easily seen that $P^* \leq P(t) \leq P(0)$ for $t \geq 0$. To show that in this case $\liminf_{t\to\infty} P_i > 0$ for $i = 1, 2$ as well, we argue as follows: From the proof of Lemma 1.4.2 we see that for any arbitrary positive constant σ_2 and for $P^* \leq P(t) \leq P(0)$,

$$\Phi_2'(t) \leq (\sigma_2 m_2(P) - m_1(P))\left(\frac{\beta_1(P)}{m_1(P)} - 1\right).$$

Choose σ_2 large enough such that $\sigma_2 m_2(P) - m_1(P) \geq 0$ for $P \in [P^*, P(0)]$. Then $\Phi_2'(t) \leq 0$, which yields $P_2^{\sigma_i}(t)/P_1(t) \leq \Phi_2(0)$. Hence if $\liminf_{t\to\infty} P_1(t) = 0$, then $\liminf_{t\to\infty} P_2(t) = 0$.

Using a similar argument, it follows that if $\liminf_{t\to\infty} P_2(t) = 0$, then $\liminf_{t\to\infty} P_1(t) = 0$. Since $P^* \leq P(t) \leq P(0)$, we obtain $\liminf_{t\to\infty} P_i(t) > 0$, $i = 1, 2$, and therefore both subpopulations should survive.

Although we have shown that in the case of two equal largest ratios, both subpopulations survive, the exact asymptotic behavior of this two-ecotype system remains complicated and may depend on the initial conditions $P_i(0)$, $i = 1, 2$. For example, suppose that $\beta_1(P) = \beta_2(P) = 1$ and $m_1(P) = m_2(P) = P$, then system (1.5.1) reduces to the following system:

$$P_i' = (1 - P)P_i, \quad P_i(0) > 0, \quad i = 1, 2.$$

Note that the equilibrium of this system satisfies $\hat{P}_1 + \hat{P}_2 = \hat{P} = 1$. Hence, there is an infinite number of equilibrium points with $0 \leq \hat{P}_1 \leq 1$ being arbitrary and $\hat{P}_2 = 1 - \hat{P}_1$. Adding the two equations, one can easily find that P satisfies a logistic dynamics and $P \to 1$ as $t \to \infty$. Furthermore, dividing the two equations, one has $dP_1/dP_2 = P_1/P_2$, and therefore $P_1/P_2 = P_1(0)/P_2(0)$. From this, one can see that $P_1 \to P_1(0)/P(0)$ and $P_2 \to P_2(0)/P(0)$ as $t \to \infty$.

In the open reproduction case, if the k^{th} $(1 \leq k \leq n)$ node in the graph associated with the matrix $[\gamma_{i,j}]$ is not connected to any other node, that is, $\gamma_{k,k} = 1$ and $\gamma_{k,j} = 0$ for $j = 1, \ldots, k-1, k+1, \ldots, n$, then the k^{th} subpopulation may become extinct. To show this, noticing $\sum_{i=1}^{n} \gamma_{i,k} = 1$, it follows that $\gamma_{i,k} = 0$ for $i = 1, \ldots, k-1, k+1, \ldots, n$. Let $\tilde{P} = \sum_{i=1}^{k-1} P_i + \sum_{i=k+1}^{n} P_i$. Summing (1.4.1) over the indices $i = 1, \ldots, k-1, k+1, \ldots, n$, we find

$$\begin{aligned}
\tilde{P}' &= \sum_{\substack{i=1 \\ i\neq k}}^{n} \left(\sum_{\substack{j=1 \\ i\neq k}}^{n} \gamma_{j,i}\beta_i(P) - m_i(P) \right) P_i \\
&= \sum_{\substack{i=1 \\ i\neq k}}^{n} (\beta_i(P) - m_i(P))P_i \\
&\geq (\underline{\beta}(P) - \overline{m}(P))\tilde{P},
\end{aligned} \tag{1.5.2}$$

where $\underline{\beta} = \min\{\beta_1, \ldots, \beta_{k-1}, \beta_{k+1}, \ldots, \beta_n\}$ and $\overline{m} = \max\{m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n\}$.

As before, we impose an assumption on the ratio of the reproduction and mortality rates.

(H8) For any $P \in I_0 = [\underline{P}, \overline{P}]$, $\dfrac{\underline{\beta}(P)}{\overline{m}(P)} > \dfrac{\beta_k(P)}{m_k(P)}$ and $\underline{\beta}(P) \geq \overline{m}(P)$.

As in the proof of Lemma 7, we choose ε ($0 < \varepsilon < \bar{\varepsilon}$) small enough so that for any $P \in I_\varepsilon \subset I_{\bar{\varepsilon}}$

$$(\sigma m_k(P) - \overline{m}(P)) \left(\frac{\underline{\beta}(P)}{\overline{m}(P)} - 1 \right) \leq \frac{1}{2} \sigma m_k(P) \left(\frac{\underline{\beta}(P)}{\overline{m}(P)} - \frac{\beta_k(P)}{m_k(P)} \right),$$

where $\sigma = \min\limits_{P \in I_{\bar{\varepsilon}}} \dfrac{\overline{m}(P)}{m_k(P)}$. We then introduce an auxiliary function $\Phi(t) = \dfrac{P_k^\sigma(t)}{\tilde{P}(t)}$.

Φ satisfies that for $t \geq t_\varepsilon^*$,

$$\begin{aligned}
\Phi' &= \frac{\sigma P_k^{\sigma-1} P_k' \tilde{P} - P_k^\sigma \tilde{P}'}{\tilde{P}^2} \\
&\leq \frac{\sigma P_k^\sigma \left(\beta_k(P) - m_k(P)\right) \tilde{P} - P_k^\sigma \left(\underline{\beta}(P) - \overline{m}(P)\right) \tilde{P}}{\tilde{P}^2} \\
&= \left[\sigma \left(\beta_k(P) - m_k(P)\right) - \left(\underline{\beta}(P) - \overline{m}(P)\right)\right] \Phi \\
&= \left[\sigma m_k(P) \left(\frac{\beta_k(P)}{m_k(P)} - 1 \right) - \overline{m}(P) \left(\frac{\underline{\beta}(P)}{\overline{m}(P)} - 1 \right)\right] \Phi \qquad (1.5.3) \\
&= \left[(\sigma m_k(P) - \overline{m}(P)) \left(\frac{\underline{\beta}(P)}{\overline{m}(P)} - 1 \right) - \sigma m_k(P) \left(\frac{\underline{\beta}(P)}{\overline{m}(P)} - \frac{\beta_k(P)}{m_k(P)} \right)\right] \Phi \\
&\leq -\frac{1}{2} \sigma m_k(P) \left(\frac{\underline{\beta}(P)}{\overline{m}(P)} - \frac{\beta_k(P)}{m_k(P)} \right) \Phi \\
&\leq -\lambda \Phi
\end{aligned}$$

with $\lambda > 0$. Integrating (1.5.3) from t_ε^* to t then yields

$$\Phi(t) \leq \Phi(t_\varepsilon^*) e^{-\lambda(t - t_\varepsilon^*)},$$

which implies $\Phi(t) \to 0$, and hence $P_k(t) \to 0$ as $t \to \infty$. Clearly, if the remaining $n-1$ ecotypes satisfy (H7) (with n replaced by $n-1$), then using previous arguments one can see that they coexist. Thus we have the following result.

Theorem 1.5.1. If the k^{th} subpopulation satisfies $\gamma_{k,k} = 1$, $\gamma_{k,j} = 0$ for $j = 1, \ldots, k-1, k+1, \ldots, n$ and hypothesis (H8) holds, then it will eventually become extinct. Moreover, if the other $n-1$ ecotypes satisfy (H7), they will coexist.

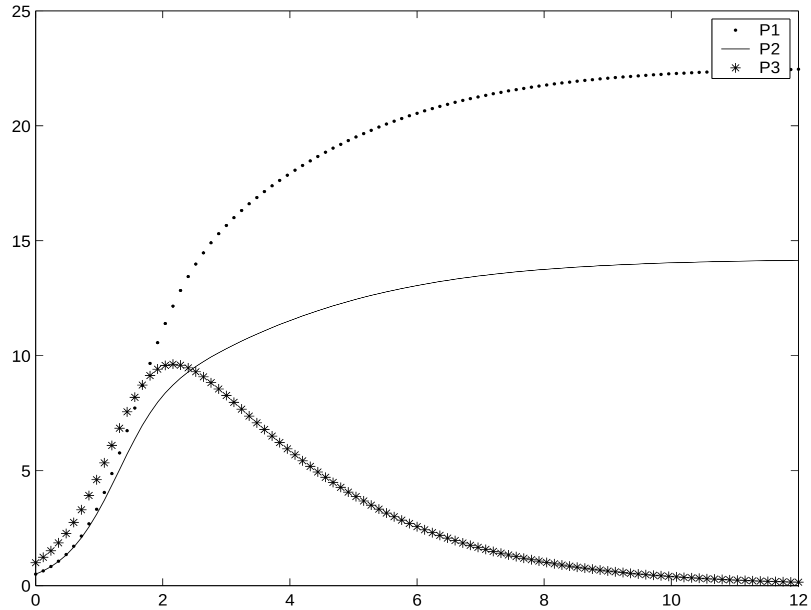

Figure 1.1: Extinction of P_3.

The next numerical example illustrates that assumption (H8) is sufficient but not necessary for the extinction of the k^{th} subpopulation. In this example, we let $n = 3$, $\beta_1 = 2.7$, $\beta_2 = 2.4$ and $\beta_3 = 2.1$. We choose the mortality functions as $m_1 = 0.054P$, $m_2 = 0.096P$ and $m_3 = 0.07P$, while the probabilities $\gamma_{i,j}$ are selected as $\gamma_{1,1} = 0.4$, $\gamma_{1,2} = 0.6$, $\gamma_{1,3} = 0$, $\gamma_{2,1} = 0.6$, $\gamma_{2,2} = 0.4, \gamma_{2,3} = 0, \gamma_{3,1} = 0$, $\gamma_{3,2} = 0$ and $\gamma_{3,3} = 1$. In Figure 1, we present the solution to this system of differential equations with the initial conditions $P_1(0) = P_2(0) = 0.5$ and $P_3(0) = 1$. The figure shows that P_3 becomes extinct although it can be easily verified that hypothesis (H8) does not hold for this example.

Our last numerical example indicates that if (H8) is not satisfied, then the k^{th} subpopulation may possibly survive while the subpopulations $P_1, \ldots, P_{k-1}, P_{k+1}, \ldots, P_n$ can become extinct. In this example, we still let $n = 3$, $\beta_1 = 2.7$, $\beta_2 = 2.4$ and $\beta_3 = 2.1$. We choose the mortality functions as $m_1 = 0.054P$, $m_2 = 0.1091P$ and $m_3 = 0.0583P$, while the probabilities $\gamma_{i,j}$ are selected as $\gamma_{1,1} = 0.1$, $\gamma_{1,2} = 0.9$, $\gamma_{1,3} = 0$, $\gamma_{2,1} = 0.9$, $\gamma_{2,2} = 0.1, \gamma_{2,3} = 0, \gamma_{3,1} = 0$, $\gamma_{3,2} = 0$ and $\gamma_{3,3} = 1$. In Figure 2, we present the numerical results for the initial conditions $P_1(0) = P_2(0) = 0.5$ and $P_3(0) = 1$. The figure shows that P_3

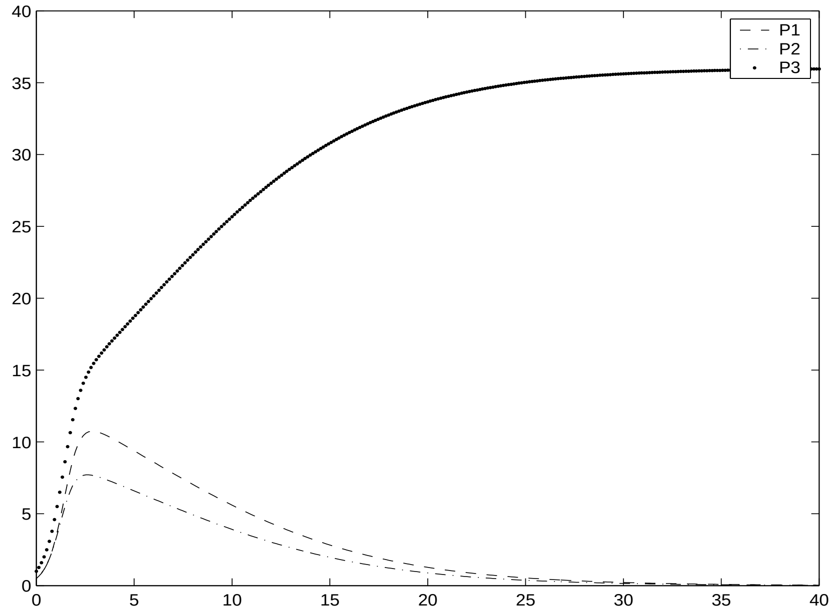

Figure 1.2: Survival of P_3 and extinction of P_1 and P_2.

survives while P_1 and P_2 die out. It is worth pointing out that even though P_1 dies out, $\dfrac{\beta_1}{m_1} > \dfrac{\beta_3}{m_3}$ for this choice of functions.

Bibliography

[1] A.S. Ackleh and L.J.S. Allen, Competitive exclusion and coexistence for pathogens in an epidemic model with variable population size, *J. Math. Biol.* **47** (2003), 153-168.

[2] A.S. Ackleh, D.F. Marshall, B.G. Fitzpatrick and H.E. Heatherly, Survival of the fittest in a generalized logistic model, *Math. Models Methods Appl. Sci.* **9** (1999), 1379-1391.

[3] A.S. Ackleh, D.F. Marshall and H.E. Heatherly, Extinction in a generalized Lotka-Volterra predator-prey model, *J. Appl. Math. Stochastic Anal.* **13** (2000), 287-297.

[4] S. Ahmad, Extinction of species in nonautonomous Lotka-Volterra systems, *Proc. Amer. Math. Soc.* **127** (1999), 2905-2910.

[5] H.T. Banks, Some remarks on estimation techniques for size-structured population models, *Frontiers of Theoretical Biology, (S. Levin, ed.)*, Springer-Verlag, Berlin, Lecture Notes in Biomathematics, **100** (1994), 609-623.

[6] H.T. Banks, L.W. Botsford, F. Kappel and C. Wang, Modeling and estimation in size-structured population models, *Proceedings 2nd Course on Mathematical Ecology (Trieste, 1986)*, World Press, Singapore, 1988, 521-541.

[7] H.T. Banks and B.G. Fitzpatrick, Estimation of growth rate distributions in size-structured population models, *Quart. Appl. Math.* **49** (1991), 215-235.

[8] H.T. Banks, B.G. Fitzpatrick, L.K. Potter and Y. Zhang, Estimation of probability distribution for individual parameters using aggregate population observation, *Stochastic Analysis, Control, Optimization and Applications (W. Mceneaney, G. Yin, Q. Zhang, eds.)*, Birkhäuser, 1998, 353-371.

[9] L.W. Botsford, B. Vandracek, T. Wainwright, A. Linden, R. Kope, D. Reed and J.J. Cech, Population development of the mosquitofish, *Gambusia Affinis*, in rice fields, *Environ. Biol. Fishes* **20** (1987), 143-154.

[10] H.J. Bremermann and H.R. Thieme, A competitive exclusion principle for pathogen virulence, *J. Math. Biol.* **27** (1989), 179-190.

[11] A. Calsina and J. Saldaña, A model of physiologically structured population dynamics with a nonlinear individual growth rate, *J. Math. Biol.* **33** (1995), 335-364.

[12] C. Castillo-Chavez, W. Huang and J. Li, Competitive exclusion in gonorrhea models and other sexually transmitted diseases, *SIAM J. Appl. Math.* **56** (1996), 494–508.

[13] C. Castillo-Chavez, W. Huang and J. Li, Competitive exclusion and coexistence of multiple strains in an SIS STD model, *SIAM J. Appl. Math.* **59** (1999), 1790-1811.

[14] Z. Feng and J.X. Velasco-Hernández, Competitve exclusion in a vector-host model for the dengue fever, *J. Math. Biol.* **35** (1997), 523-544.

[15] B.G. Fitzpatrick, Modeling and estimation problems for structured heterogeneous populations, *J. Math. Anal. Appl.* **172** (1993), 73-91.

[16] B.G. Fitzpatrick, Rate distribution modeling for structured heterogeneous populations, *Control and Estimated Parameter Systems: Nonlinear Phenomena (Vorau, 1993), Internat. Ser. Numer. Math.* Birkhäuser, Basel, **118** (1994), 131-141.

[17] S.M. Henson and T.G. Hallam, Survival of the fittest: asymptotic competitive exclusion in structured population and community models, *Nonlinear World* **1** (1994), 385-402.

[18] S. Hsu T. Hwang and Y. Kuang, Rich dynamics of a ratio-dependent one-prey two-predators model, *J. Math. Biol.* **43** (2001), 377-396.

[19] T. Lindström, Global stability of a model for competing predators: an extension of the Ardito & Ricciardi Lyapunov function, *Nonlinear Anal.* **39** (2000), 793-805.

[20] F. Montes de Oca and M.L. Zeeman, Extinction in nonautonomous competitive Lotka-Volterra systems, *Proc. Amer. Math. Soc.* **124** (1996), 3677-3687.

[21] H.L. Smith, A discrete, size-structured model of microbial growth and competition in the chemostat, *J. Math. Biol.* **34** (1996), 734-754.

[22] H.L. Smith and P. Waltman, Competition for a single limiting resource in continuous culture: the variable-yield model, *SIAM J. Appl. Math.* **54** (1994), 1113-1131.

[23] H.L. Smith and X.-Q. Zhao, Competitive exclusion in a discrete-time, size-structured chemostat model, *Discrete Contin. Dynam. Syst. Ser. B* **1** (2001), 183-191.

[24] M.L. Zeeman, Extinction in competitive Lotka-Volterra systems, *Proc. Amer. Math. Soc.* **123** (1995), 87-96.

Chapter 2

Existence-Uniqueness and Monotone Approximation for a Phytoplankton-Zooplankton Aggregation Model

In this chapter, we study a model that describes the dynamics of the phytoplankton and zooplankton prey-predator system within the context of phytoplankton aggregation. Existence-uniqueness results of the solution are established via a comparison principle and the upper-lower solution technique.

2.1 Introduction

In recent years, several studies have been devoted to the development of phytoplankton aggregation population models (see [2]-[5] and the references cited therein). In [4], an autonomous size-structured model describing the dynamics of phytoplankton aggregation was investigated. Therein, existence and uniqueness of the solution were established via the semigroups of linear operators theoretic approach. Furthermore, by using the Trotter-Kato technique, a numerical method based on approximating semigroups was developed to compute the solution. In [1], an inverse problem of identifying parameters in the model developed in [4] was studied. In [3], this model was extended to a two-dimensional setting, in which tracking of the aggregate densities at different depths of the surface

The results of this chapter have been submitted for publication.

layer of the ocean was modeled. A finite difference method approximating the solution was developed, and the convergence of this approximation to a unique bounded variation solution was established. In [2], a nonautonomous version of the model discussed in [4] was considered with an unbounded domain for the particle size, and existence-uniqueness results were established via the construction of monotone sequences.

Modeling efforts have revealed the importance of aggregation in structuring marine phytoplankton communities. A common conclusion of such studies is that aggregation limits phytoplankton densities by combining small single cells into large, faster-settling aggregates. However, the relative importance of grazing and aggregation on influencing the dynamics of phytoplankton is very little documented in the literature. In fact, we only know of article [6] that discusses grazing and aggregation effect on phytoplankton. Therein, the authors demonstrate via simulations that the determinants of roles of aggregation and predation in phytoplankton transport are complex and depend on biophysical, chemical and biological forces. Such simulations also show that grazing can make an indirect impact on the distribution of aggregate volumes and consequently on the sinking rates of all aggregates. In this chapter, we consider a generalization of the model presented in [6]. In particular, we model cell division within aggregates and allow the parameters to be time-dependent. The model considered here is as follows:

$$
\begin{aligned}
&u_t + (g(x,t)u)_x + m(x,t,Z,\varphi(u))u \\
&\qquad = \frac{1}{2}\int_0^x \beta(x-y,y)u(x-y,t)u(y,t)dy \\
&\qquad\quad - \int_0^\infty \beta(x,y)u(x,t)u(y,t)dy && (x,t)\in(0,\infty)\times(0,T),\\
&\frac{dZ}{dt} = [f_1(t,\varphi(u)) - f_2(t,Z)]Z && t\in(0,T),\\
&g(0,t)u(0,t) = \int_0^\infty \gamma(y,t,Z(t))u(y,t)dy && t\in(0,T),\\
&u(x,0) = u_0(x) && x\in[0,\infty),\\
&Z(0) = Z_0,
\end{aligned}
\tag{2.1.1}
$$

where $\varphi(u) = \int_0^\infty w(x)u(x,t)dx$ with $0 \leq w(x) \leq 1$. The first differential equation models the dynamics of phytoplankton population with aggregation between individual cells. The function $u(x,t)$ represents the density of phytoplankton aggregates of size x at time t. The function $Z(t)$ describes the density of the predator (zooplankton) at time t. The parameter g represents the growth rate of an aggregate of size x due to cell division within it, while the function m represents the proportion of aggregates that is being lost due to factors such as sinking of the surface layer of the ocean (see [6]). The parameter $\beta(x,y)$ is the likelihood of an aggregate of size x sticking to an aggregate of size y. The function γ is the number of single cells that break off an aggregate of size x and join the single cell population. The first term on the right-hand side of equation $(2.1.1)_1$ expresses the rate at which collisions occur to form new particles within the sizes x and $x + dx$, while the second term expresses the rate at which collisions cause particles to be lost from the same size interval. The second differential equation describes the dynamics of zooplankton. The function f_1 represents the growth rate that depends on phytoplankton availability, and f_2 describes the mortality rate of zooplankton, which may depend on Z due to competition. For example, in [6] $f_1 = k_1\varphi/(\varphi + k_2)$ and $f_2 = k_3 Z^r$, where k_i $(i = 1, 2, 3)$ is positive constant and r is taken to be either 0 for a first order grazing process or 1 for a logistic representation.

To the best of our knowledge, comparison principle and approximation method based on monotone sequences have not been developed for the prey-predator system (2.1.1). Therefore, in this chapter we undertake such a task to establish the existence and uniqueness of the solution to (2.1.1). For simplicity, let $D_T = (0,\infty) \times (0,T)$ and $C^1_{0,r}(D_T) = \{\psi \in C^1(D_T) : \exists\, x_\psi \in (0,\infty) \text{ such that } \psi \equiv 0 \text{ for } x \geq x_\psi\}$. We define the solution of (2.1.1) as follows:

Definition 2.1.1. $(u(x,t), Z(t))$ is called a solution of (2.1.1) on D_T, if all the following hold:

(i) $u \in L^\infty((0,T); L^1(0,\infty) \cap L^\infty(0,\infty))$; $Z \in L^\infty(0,T)$.

(ii) $u(x,0) = u_0(x)$ a.e. *in* $(0,\infty)$; $Z(0) = Z_0$.

(iii) For each $t \in (0,T)$ and every nonnegative $\xi \in C^1_{0,r}(D_T)$,

$$\begin{aligned}
&\int_0^\infty u(x,t)\xi(x,t)dx \\
&\quad = \int_0^\infty u(x,0)\xi(x,0)dx \\
&\qquad + \int_0^t \xi(0,s)\int_0^\infty \gamma(x,s,Z(s))u(x,s)dxds \\
&\qquad + \int_0^t \int_0^\infty [\xi_s(x,s) + g(x,s)\xi_x(x,s)]u(x,s)dxds \\
&\qquad + \int_0^t \int_0^\infty \xi(x,s)(\mathcal{F}u)(x,s)dxds \\
&\qquad - \int_0^t \int_0^\infty \xi(x,s)\int_0^\infty \beta(x,y)u(x,s)u(y,s)dydxds \\
&\qquad - \int_0^t \int_0^\infty \xi(x,s)m(x,s,Z(s),\varphi(u))u(x,s)dxds,
\end{aligned} \tag{2.1.2}$$

$$Z(t) = Z(0) + \int_0^t \left[f_1(s,\varphi(u)) - f_2(s,Z(s))\right] Z(s)ds,$$

where

$$(\mathcal{F}u)(x,t) = \frac{1}{2}\int_0^x \beta(x-y,y)u(x-y,t)u(y,t)dy. \tag{2.1.3}$$

Our arguments here are in the spirit of those used for a coagulation model [2]. Such arguments are based on a novel definition of coupled upper and lower solutions, the establishment of a comparison principle and the construction of monotone sequences of upper and lower solutions which will lead to the existence of the solution by passing to the limit. It is worth mentioning that in [8, 9] the semigroups of linear operators theoretic approach was used to show the existence-uniqueness of the solution to an autonomous coagulation model. These results were extended to a time-dependent coagulation kernel using evolution operator theory [10]. However, the semigroup approach does not apply to nonautonomous models. On the other hand, the evolution operator theory may fail due to the presence of the nonlocal boundary condition $(2.1.1)_3$ (see [7] for discussion).

We organize this chapter as follows. In Section 2.2, we give the definition of upper and lower solutions and establish a comparison principle. In Section 2.3, we construct two monotone sequences of upper and lower solutions and show their convergence to the unique local solution of (2.1.1).

2.2 Comparison principle

Throughout our discussion, we impose the following hypotheses.

(H1) $g(x,t)$ is continuously differentiable on $(0,\infty)\times(0,T)$, $g(x,t)>0$ for $(x,t)\in[0,\infty)\times[0,T]$ and $\lim_{x\to\infty} g(x,t)=0$ for $t\in[0,T]$.

(H2) $m(x,t,Z,\varphi)$ (≥ 0) is continuous on $[0,\infty)\times[0,T]\times[0,\infty)\times[0,\infty)$ with $\|m\|_\infty<\infty$. Furthermore, m_Z, m_φ exist and satisfy $0\leq m_Z, m_\varphi<\infty$.

(H3) $\beta(x,y)$ (≥ 0) is continuous on $[0,\infty)\times[0,\infty)$ with $\|\beta\|_\infty<\infty$.

(H4) $f_1(t,\varphi)$ (≥ 0) is continuous on $[0,T]\times[0,\infty)$. Furthermore, $f_{1\varphi}$ exists and satisfies $0\leq f_{1\varphi}<\infty$.

(H5) $f_2(t,Z)$ (≥ 0) is continuous on $[0,T]\times[0,\infty)$. Furthermore, f_{2Z} exists and satisfies $0\leq f_{2Z}<\infty$.

(H6) $\gamma(x,t,Z)$ (≥ 0) is continuous on $[0,\infty)\times[0,T]\times[0,\infty)$ with $\|\gamma\|_\infty<\infty$. Furthermore, γ_Z exists and satisfies $-\infty<\gamma_Z\leq 0$.

(H7) $u_0(x)\geq 0$ on $[0,\infty)$ and $u_0\in L^1(0,\infty)\cap L^\infty(0,\infty)$.

(H8) $Z_0\geq 0$.

We begin with the definition of a pair of coupled upper and lower solutions of problem (2.1.1).

Definition 2.2.1. A pair of functions $(\overline{u}(x,t), \overline{Z}(t))$ and $(\underline{u}(x,t), \underline{Z}(t))$ are called an upper solution and a lower solution of (2.1.1) on D_T, respectively, if all the following hold:

(i) $\overline{u}, \underline{u} \in L^\infty((0,T); L^1(0,\infty) \cap L^\infty(0,\infty))$; $\overline{Z}, \underline{Z} \in L^\infty(0,T)$.

(ii) $\overline{u}(x,0) \geq u_0(x) \geq \underline{u}(x,0)$ a.e. in $(0,\infty)$; $\overline{Z}(0) \geq Z_0 \geq \underline{Z}(0)$.

(iii) For each $t \in (0,T)$, every nonnegative $\xi \in C^1_{0,r}(D_T)$ and every nonnegative $\eta \in C^1(0,T)$,

$$\begin{aligned}
&\int_0^\infty \overline{u}(x,t)\xi(x,t)dx \\
&\geq \int_0^\infty \overline{u}(x,0)\xi(x,0)dx \\
&\quad + \int_0^t \xi(0,s) \int_0^\infty \gamma(x,s,\underline{Z}(s))\overline{u}(x,s)dxds \\
&\quad + \int_0^t \int_0^\infty [\xi_s(x,s) + g(x,s)\xi_x(x,s)]\overline{u}(x,s)dxds \\
&\quad + \int_0^t \int_0^\infty \xi(x,s)(\mathcal{F}\overline{u})(x,s)dxds \qquad (2.2.1) \\
&\quad - \int_0^t \int_0^\infty \xi(x,s) \int_0^\infty \beta(x,y)\overline{u}(x,s)\underline{u}(y,s)dydxds \\
&\quad - \int_0^t \int_0^\infty \xi(x,s)m(x,s,\underline{Z}(s),\varphi(\underline{u}))\overline{u}(x,s)dxds,
\end{aligned}$$

$$\begin{aligned}
\overline{Z}(t)\eta(t) \geq \overline{Z}(0)\eta(0) &+ \int_0^t \overline{Z}(s)\eta'(s)ds \\
&+ \int_0^t [f_1(s,\varphi(\overline{u})) - f_2(s,\underline{Z}(s))]\,\overline{Z}(s)\eta(s)ds;
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty \underline{u}(x,t)\xi(x,t)dx \\
&\leq \int_0^\infty \underline{u}(x,0)\xi(x,0)dx \\
&\quad + \int_0^t \xi(0,s) \int_0^\infty \gamma(x,s,\overline{Z}(s))\underline{u}(x,s)dxds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_0^\infty [\xi_s(x,s)+g(x,s)\xi_x(x,s)]\underline{u}(x,s)dxds \\
&+\int_0^t\int_0^\infty \xi(x,s)(\mathcal{F}\underline{u})(x,s)dxds \\
&-\int_0^t\int_0^\infty \xi(x,s)\int_0^\infty \beta(x,y)\underline{u}(x,s)\overline{u}(y,s)dydxds \\
&-\int_0^t\int_0^\infty \xi(x,s)m(x,s,\overline{Z}(s),\varphi(\overline{u}))\underline{u}(x,s)dxds,
\end{aligned} \tag{2.2.2}
$$

$$
\begin{aligned}
\underline{Z}(t)\eta(t) \le \underline{Z}(0)\eta(0) &+ \int_0^t \underline{Z}(s)\eta'(s)ds \\
&+\int_0^t \left[f_1(s,\varphi(\underline{u}))-f_2(s,\overline{Z}(s))\right]\underline{Z}(s)\eta(s)ds.
\end{aligned}
$$

Based on such a definition, we now establish the following comparison principle.

Theorem 2.2.2. Suppose that (H1)-(H8) hold. Let $(\overline{u},\overline{Z})$ and $(\underline{u},\underline{Z})$ be a nonnegative upper solution and a nonnegative lower solution of (2.1.1), respectively. Then $\overline{u}\ge\underline{u}$ a.e. in D_T and $\overline{Z}\ge\underline{Z}$ in $(0,T)$.

Proof. Let $v=\underline{u}-\overline{u}$ and $Y=\underline{Z}-\overline{Z}$. Choose $\xi\in C^1_{0,r}((0,n)\times(0,T))$, where $C^1_{0,r}((0,n)\times(0,T))=\{\psi\in C^1((0,n)\times(0,T)) : \exists\, x_\psi\in(0,n)$ such that $\psi\equiv 0$ for $x\ge x_\psi\}$. Then $v(x,0)=\underline{u}(x,0)-\overline{u}(x,0)\le 0$ a.e. in $(0,\infty)$ and $Y(0)=\underline{Z}(0)-\overline{Z}(0)\le 0$. Furthermore, v and Y satisfy

$$
\begin{aligned}
&\int_0^\infty v(x,t)\xi(x,t)dx \\
&\quad\le \int_0^\infty v(x,0)\xi(x,0)dx \\
&\qquad+\int_0^t \xi(0,s)\int_0^\infty \gamma(x,s,\overline{Z}(s))v(x,s)dxds \\
&\qquad+\int_0^t\int_0^\infty [\xi_s(x,s)+g(x,s)\xi_x(x,s)]v(x,s)dxds \\
&\qquad+\int_0^t\int_0^\infty \xi(x,s)[(\mathcal{F}\underline{u})(x,s)-(\mathcal{F}\overline{u})(x,s)]dxds
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^t \int_0^\infty \xi(x,s)v(x,s)\int_0^\infty \beta(x,y)\overline{u}(y,s)dydxds\\
&+\int_0^t \int_0^\infty \xi(x,s)\overline{u}(x,s)\int_0^\infty \beta(x,y)v(y,s)dydxds\\
&-\int_0^t \int_0^\infty \xi(x,s)m(x,s,\underline{Z}(s),\varphi(\underline{u}))v(x,s)dxds\\
&-\int_0^t \xi(0,s)\int_0^\infty \gamma_Z(x,s,Z_1(s))\overline{u}(x,s)Y(s)dxds\\
&+\int_0^t \int_0^\infty \xi(x,s)m_Z(x,s,Z_2(s),\varphi(\underline{u}))\underline{u}(x,s)Y(s)dxds\\
&+\int_0^t \int_0^\infty \xi(x,s)m_\varphi(x,s,\overline{Z}(s),\theta_1(s))\underline{u}(x,s)\int_0^\infty w(y)v(y,s)dydxds,
\end{aligned} \tag{2.2.3}
$$

$$
\begin{aligned}
Y(t)\eta(t) \le Y(0)\eta(0) &+ \int_0^t Y(s)\eta'(s)ds\\
&+\int_0^t \overline{Z}(s)f_{1\varphi}(s,\theta_2(s))\eta(s)\int_0^\infty w(x)v(x,s)dxds\\
&+\int_0^t \left[f_1(s,\varphi(\underline{u})) - f_2(s,\overline{Z}(s)) + f_{2Z}(s,Z_3(s))\overline{Z}(s)\right]Y(s)\eta(s)ds
\end{aligned}
$$

with Z_1, Z_2 and Z_3 between $\underline{Z}$ and $\overline{Z}$, θ_1 and θ_2 between $\varphi(\underline{u})$ and $\varphi(\overline{u})$.

Upon manipulation, it is easy to see that

$$
\begin{aligned}
&\int_0^t \int_0^\infty \xi(x,s)[(\mathcal{F}\underline{u})(x,s) - (\mathcal{F}\overline{u})(x,s)]dxds\\
&\quad = \frac{1}{2}\int_0^t \int_0^\infty v(y,s)\int_0^\infty \xi(y+z,s)\beta(z,y)\underline{u}(z,s)dzdyds\\
&\quad\quad + \frac{1}{2}\int_0^t \int_0^\infty \overline{u}(y,s)\int_0^\infty \xi(y+z,s)\beta(z,y)v(z,s)dzdyds.
\end{aligned} \tag{2.2.4}
$$

Let $\xi(x,t) = e^{\lambda t}\zeta(x,t)$, where $\zeta \in C^1_{0,r}((0,n)\times(0,T))$ and λ (>0) is chosen so that $\lambda - \int_0^\infty \beta(x,y)\overline{u}(y,t)dy - m(x,t,\underline{Z}(t),\varphi(\underline{u})) \ge 0$ on D_T. Then $(2.2.3)_1$ becomes

$$
\begin{aligned}
&e^{\lambda t}\int_0^\infty v(x,t)\zeta(x,t)dx\\
&\le \int_0^\infty v(x,0)\zeta(x,0)dx + \int_0^t e^{\lambda s}\zeta(0,s)\int_0^\infty \gamma(x,s,\overline{Z}(s))v(x,s)dxds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_0^\infty v(x,s)e^{\lambda s}[\zeta_s(x,s)+g(x,s)\zeta_x(x,s)]dxds\\
&+\int_0^t\int_0^\infty v(x,s)e^{\lambda s}\zeta(x,s)\left(\lambda-\int_0^\infty \beta(x,y)\overline{u}(y,s)dy-m(x,s,\underline{Z}(s),\varphi(\underline{u}))\right)dxds\\
&+\frac{1}{2}\int_0^t\int_0^\infty v(y,s)\int_0^\infty e^{\lambda s}\zeta(y+z,s)\beta(z,y)\underline{u}(z,s)dzdyds\\
&+\frac{1}{2}\int_0^t\int_0^\infty \overline{u}(y,s)\int_0^\infty e^{\lambda s}\zeta(y+z,s)\beta(z,y)v(z,s)dzdyds\\
&+\int_0^t\int_0^\infty e^{\lambda s}\zeta(x,s)\overline{u}(x,s)\int_0^\infty \beta(x,y)v(y,s)dydxds\\
&-\int_0^t e^{\lambda s}\zeta(0,s)\int_0^\infty \gamma_Z(x,s,Z_1(s))\overline{u}(x,s)Y(s)dxds\\
&+\int_0^t\int_0^\infty e^{\lambda s}\zeta(x,s)m_Z(x,s,Z_2(s),\varphi(\underline{u}))\underline{u}(x,s)Y(s)dxds\\
&+\int_0^t\int_0^\infty e^{\lambda s}\zeta(x,s)m_\varphi(x,s,\overline{Z}(s),\theta_1(s))\underline{u}(x,s)\int_0^\infty w(y)v(y,s)dydxds.
\end{aligned}
\tag{2.2.5}
$$

We now set up a backward problem as follows:

$$
\begin{aligned}
&\zeta_s+g\zeta_x=0 && 0<x<n, \quad 0<s<t,\\
&\zeta(n,s)=0 && \qquad\qquad\quad 0<s<t,\\
&\zeta(x,t)=\chi(x) && 0\le x\le n.
\end{aligned}
$$

Here $\chi\in C_0^\infty(0,n),\ 0\le\chi\le 1$.

The existence of $\zeta\in C^1_{0,r}((0,n)\times(0,T))$ follows from the fact that by the variable change $\tau=t-s$, the above problem can be written into

$$
\begin{aligned}
&\zeta_\tau-g\zeta_x=0 && 0<x<n, \quad 0<\tau<t,\\
&\zeta(n,\tau)=0 && \qquad\qquad\quad 0<\tau<t,\\
&\zeta(x,0)=\chi(x) && 0\le x\le n.
\end{aligned}
$$

Note that the initial and boundary values for ζ imply that $0\le\zeta\le 1$ on $(0,n)\times(0,T)$.

Substituting such a ζ in (2.2.5), we have

$$
\begin{aligned}
&\int_0^n v(x,t)\chi(x)dx\\
&\qquad\le\int_0^\infty v^+(x,0)dx+\nu\int_0^t\int_0^\infty v^+(x,s)dxds
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^t e^{-\lambda(t-s)}\zeta(0,s)\int_0^\infty \gamma_Z(x,s,Z_1(s))\overline{u}(x,s)Y(s)dxds \\
&+ \int_0^t \int_0^\infty e^{-\lambda(t-s)}\zeta(x,s)m_Z(x,s,Z_2(s),\varphi(\underline{u}))\underline{u}(x,s)Y(s)dxds,
\end{aligned} \tag{2.2.6}
$$

where

$$
\begin{aligned}
\nu = \sup_{\overline{D_T}} \Bigg[&\gamma(x,t,\overline{Z}(t)) + \left(\lambda - \int_0^\infty \beta(x,y)\overline{u}(y,t)dy - m(x,t,\underline{Z}(t),\varphi(\underline{u}))\right) \\
&+ \frac{1}{2}\int_0^\infty \beta(z,x)\underline{u}(z,t)dz + \frac{3}{2}\|\beta\|_\infty \int_0^\infty \overline{u}(x,t)dx \\
&+ \int_0^\infty m_\varphi(x,t,\overline{Z}(t),\theta_1(t))\underline{u}(x,t)dx \Bigg].
\end{aligned}
$$

From the initial data for v, we then find

$$
\begin{aligned}
&\int_0^n v(x,t)\chi(x)dx \\
&\quad \le \nu \int_0^t \int_0^\infty v^+(x,s)dxds \\
&\quad\quad - \int_0^t e^{-\lambda(t-s)}\zeta(0,s)\int_0^\infty \gamma_Z(x,s,Z_1(s))\overline{u}(x,s)Y(s)dxds \\
&\quad\quad + \int_0^t \int_0^\infty e^{-\lambda(t-s)}\zeta(x,s)m_Z(x,s,Z_2(s),\varphi(\underline{u}))\underline{u}(x,s)Y(s)dxds.
\end{aligned}
$$

Since this inequality holds for every χ, we can choose a sequence $\{\chi_k\}$ on $(0,n)$ converging to

$$
\chi = \begin{cases} 1 & \text{if } v(x,t) > 0, \\ 0 & \text{otherwise.} \end{cases}
$$

Consequently, we have

$$
\begin{aligned}
&\int_0^n v^+(x,t)dx \\
&\quad \le \nu \int_0^t \int_0^\infty v^+(x,s)dxds \\
&\quad\quad - \int_0^t e^{-\lambda(t-s)}\zeta(0,s)\int_0^\infty \gamma_Z(x,s,Z_1(s))\overline{u}(x,s)Y(s)dxds \\
&\quad\quad + \int_0^t \int_0^\infty e^{-\lambda(t-s)}\zeta(x,s)m_Z(x,s,Z_2(s),\varphi(\underline{u}))\underline{u}(x,s)Y(s)dxds.
\end{aligned}
$$

Since ν is independent of n, letting $n \to \infty$, we find

$$\begin{aligned}
&\int_0^\infty v^+(x,t)dx \\
&\quad \le \nu \int_0^t \int_0^\infty v^+(x,s)dxds \\
&\quad\ - \int_0^t e^{-\lambda(t-s)}\zeta(0,s)\int_0^\infty \gamma_Z(x,s,Z_1(s))\overline{u}(x,s)Y(s)dxds \\
&\quad\ + \int_0^t \int_0^\infty e^{-\lambda(t-s)}\zeta(x,s)m_Z(x,s,Z_2(s),\varphi(\underline{u}))\underline{u}(x,s)Y(s)dxds.
\end{aligned} \tag{2.2.7}$$

On the other hand, let $\eta(t) = e^{\mu t}$, $\alpha(t) = \mu + f_1(t,\varphi(\underline{u})) - f_2(t,\overline{Z}(t)) + f_{2Z}(t,Z_3(t))\overline{Z}(t)$, and $\delta(t) = \overline{Z}(t) f_{1\varphi}(t,\theta_2(t))$, where μ (> 0) is chosen so that $\alpha(t) \ge 0$ in $[0,T]$. In view of the initial condition for Y, by $(2.2.3)_2$ we obtain

$$Y(t)e^{\mu t} \le \int_0^t e^{\mu s}\delta(s)\int_0^\infty w(x)v^+(x,s)dxds + \int_0^t \alpha(s)Y(s)e^{\mu s}ds. \tag{2.2.8}$$

Let $\Psi(t) = \int_0^t e^{\mu s}\delta(s)\int_0^\infty w(x)v^+(x,s)dxds + \int_0^t \alpha(s)Y(s)e^{\mu s}ds$. By (2.2.8) we find that $\Psi(0) = 0$ and

$$\begin{aligned}
\Psi'(t) &\le e^{\mu t}\delta(t)\int_0^\infty w(x)v^+(x,t)dx + \alpha(t)Y(t)e^{\mu t} \\
&\le e^{\mu t}\delta(t)\int_0^\infty w(x)v^+(x,t)dx + \alpha(t)\Psi(t),
\end{aligned}$$

or equivalently,

$$e^{-\int_0^t \alpha(\tau)d\tau}\left(\Psi'(t) - \alpha(t)\Psi(t)\right) \le e^{-\int_0^t \alpha(\tau)d\tau} e^{\mu t}\delta(t)\int_0^\infty w(x)v^+(x,t)dx.$$

Integrating the above inequality from 0 to t and making use of (2.2.8), we then find

$$Y(t)e^{\mu t} \le e^{\int_0^t \alpha(\tau)d\tau}\int_0^t e^{-\int_0^s \alpha(\tau)d\tau} e^{\mu s}\delta(s)\int_0^\infty w(x)v^+(x,s)dxds.$$

Hence we have

$$Y(t) \le e^{\int_0^t \alpha(\tau)d\tau}\int_0^t \delta(s)\int_0^\infty v^+(x,s)dxds. \tag{2.2.9}$$

Substituting (2.2.9) in (2.2.7) then yields

$$\begin{aligned}
\int_0^\infty v^+(x,t)dx &\le \nu\int_0^t\int_0^\infty v^+(x,s)dxds + \kappa\int_0^t\int_0^s\int_0^\infty v^+(x,\tau)dxd\tau ds \\
&\le (\nu + \kappa T)\int_0^t\int_0^\infty v^+(x,s)dxds,
\end{aligned} \tag{2.2.10}$$

where

$$\kappa = \sup_{0\le t\le T} \delta(t) e^{\int_0^T \alpha(t)dt} \sup_{0\le t\le T} \int_0^\infty \left[-\gamma_Z(x,t,Z_1(t))\overline{u}(x,t) + m_Z(x,t,Z_2(t),\varphi(\underline{u}))\underline{u}(x,t)\right] dx.$$

Upon application of Gronwall's inequality, (2.2.10) leads to

$$\int_0^\infty v^+(x,t)dx = 0.$$

It then follows that $v \le 0$ a.e. in D_T, and by (2.2.9) $Y(t) \le 0$ in $(0,T)$. Thus, the proof is completed. □

Remark 1. From the proof of Theorem 2.2.2, it is easily seen that for $v \in L^\infty((0,T); L^1(0,\infty) \cap L^\infty(0,\infty))$ and $Y \in L^\infty(0,T)$, if $v(x,0) \le 0$ a.e. in $(0,\infty)$, $Y(0) \le 0$, and the following inequalities hold for every nonnegative $\xi \in C^1_{0,r}(D_T)$ and every nonnegative $\eta \in C^1(0,T)$,

$$\begin{aligned}
\int_0^\infty v(x,t)\xi(x,t)dx \le \int_0^\infty & v(x,0)\xi(x,0)dx \\
&+ \int_0^t \int_0^\infty [\xi_s(x,s) + g(x,s)\xi_x(x,s)]v(x,s)dxds \\
&+ \int_0^t \int_0^\infty \xi(x,s)A(x,s)v(x,s)dxds, \\
Y(t)\eta(t) \le Y(0)\eta(0) &+ \int_0^t Y(s)\eta'(s)ds + \int_0^t \rho(s)Y(s)\eta(s)ds,
\end{aligned} \tag{2.2.11}$$

where $A \in L^\infty(D_T)$ and $\rho \in L^\infty(0,T)$, then $v(x,t) \le 0$ a.e. in D_T and $Y(t) \le 0$ in $(0,T)$. Such a result will be used in Section 3.

We then establish the following uniqueness result.

Theorem 2.2.3. Suppose that (H1)-(H8) hold. Then problem (2.1.1) has at most one solution.

Proof. Suppose that $(\hat{u}, \hat{Z})$ and $(\tilde{u}, \tilde{Z})$ are two solutions of (2.1.1). Let $v(x,t) = \hat{u}(x,t) - \tilde{u}(x,t)$ and $Y(t) = \hat{Z}(t) - \tilde{Z}(t)$. Then (v, Y) satisfies

$$
\begin{aligned}
&\int_0^\infty v(x,t)\xi(x,t)dx \\
&\quad = \int_0^t \xi(0,s)\int_0^\infty \gamma(x,s,\hat{Z}(s))v(x,s)dxds \\
&\qquad + \int_0^t\int_0^\infty [\xi_s(x,s)+g(x,s)\xi_x(x,s)]v(x,s)dxds \\
&\qquad - \int_0^t\int_0^\infty v(x,s)\xi(x,s)\left(\int_0^\infty \beta(x,y)\hat{u}(y,s)dy + m(x,s,\hat{Z}(s),\varphi(\hat{u}))\right)dxds \\
&\qquad + \frac{1}{2}\int_0^t\int_0^\infty v(y,s)\int_0^\infty \xi(y+z,s)\beta(z,y)\hat{u}(z,s)dzdyds \\
&\qquad + \frac{1}{2}\int_0^t\int_0^\infty \tilde{u}(y,s)\int_0^\infty \xi(y+z,s)\beta(z,y)v(z,s)dzdyds \\
&\qquad - \int_0^t\int_0^\infty \xi(x,s)\tilde{u}(x,s)\int_0^\infty \beta(x,y)v(y,s)dydxds \\
&\qquad + \int_0^t \xi(0,s)\int_0^\infty \gamma_Z(x,s,Z_4(s))\tilde{u}(x,s)Y(s)dxds \\
&\qquad - \int_0^t\int_0^\infty \xi(x,s)m_Z(x,s,Z_5(s),\varphi(\hat{u}))\tilde{u}(x,s)Y(s)dxds \\
&\qquad - \int_0^t\int_0^\infty \xi(x,s)m_\varphi(x,s,\tilde{Z}(s),\theta_3(s))\tilde{u}(x,s)\int_0^\infty w(y)v(y,s)dydxds, \\
&Y(t) = \int_0^t \tilde{Z}(s)f_{1\varphi}(s,\theta_4(s))\int_0^\infty w(x)v(x,s)dxds \\
&\qquad + \int_0^t \left[f_1(s,\varphi(\hat{u})) - f_2(s,\hat{Z}(s)) - f_{2Z}(s,Z_6(s))\tilde{Z}(s)\right]Y(s)ds
\end{aligned}
\tag{2.2.12}
$$

with Z_4, Z_5 and Z_6 between $\hat{Z}$ and $\tilde{Z}$, θ_3 and θ_4 between $\varphi(\hat{u})$ and $\varphi(\tilde{u})$.

Choose $\xi \in C^1_{0,r}((0,n)\times(0,T))$, where ξ satisfies

$$
\begin{aligned}
&\xi_s + g\xi_x = 0 && 0<x<n, \quad 0<s<t, \\
&\xi(n,s) = 0 && \qquad\qquad 0<s<t, \\
&\xi(x,t) = \tilde{\chi}(x) && 0\le x\le n.
\end{aligned}
$$

Here $\tilde{\chi} \in C_0^\infty(0,n)$, $-1 \le \tilde{\chi} \le 1$, which implies $-1 \le \xi \le 1$.

Substituting such a ξ in $(2.2.12)_1$ and letting $\tilde{\alpha}(t) = f_1(t,\varphi(\hat{u})) - f_2(t,\hat{Z}(t)) - f_{2Z}(t,Z_6(t))\tilde{Z}(t)$ and $\tilde{\delta}(t) = \tilde{Z}(t)f_{1\varphi}(t,\theta_4(t))$ in $(2.2.12)_2$, we find that

$$\begin{aligned}
&\int_0^n v(x,t)\tilde{\chi}(x)dx \\
&\quad \le \tilde{\nu}\int_0^t\int_0^\infty |v(x,s)|dxds \\
&\quad\quad -\int_0^t\int_0^\infty \gamma_Z(x,s,Z_4(s))\tilde{u}(x,s)|Y(s)|dxds \\
&\quad\quad +\int_0^t\int_0^\infty m_Z(x,s,Z_5(s),\varphi(\hat{u}))\tilde{u}(x,s)|Y(s)|dxds,
\end{aligned}$$

$$|Y(t)| \le e^{\int_0^t |\tilde{\alpha}(\tau)|d\tau}\int_0^t \tilde{\delta}(s)\int_0^\infty |v(x,s)|dxds, \tag{2.2.13}$$

where

$$\begin{aligned}
\tilde{\nu} = \sup_{\overline{D_T}}\Bigg(&\gamma(x,t,\hat{Z}(t)) + \int_0^\infty \beta(x,y)\hat{u}(y,t)dy + m(x,t,\hat{Z}(t),\varphi(\hat{u})) \\
&+\frac{1}{2}\int_0^\infty \beta(z,x)\hat{u}(z,t)dz + \frac{3}{2}\|\beta\|_\infty\int_0^\infty \tilde{u}(x,t)dx \\
&+\int_0^\infty m_\varphi(x,t,\tilde{Z}(t),\theta_3(t))\tilde{u}(x,t)dx\Bigg).
\end{aligned}$$

Since inequality $(2.2.13)_1$ holds for every $\tilde{\chi}$, we can choose a sequence $\tilde{\chi}_k$ on $(0,n)$ converging to

$$\tilde{\chi} = \operatorname{sgn}(v(x,t)) = \begin{cases} 1 & \text{if } v(x,t) > 0, \\ 0 & \text{if } v(x,t) = 0, \\ -1 & \text{if } v(x,t) < 0. \end{cases}$$

Consequently, we have

$$\begin{aligned}
&\int_0^n |v(x,t)|dx \\
&\quad \le \tilde{\nu}\int_0^t\int_0^\infty |v(x,s)|dxds \\
&\quad\quad -\int_0^t\int_0^\infty \gamma_Z(x,s,Z_4(s))\tilde{u}(x,s)|Y(s)|dxds \\
&\quad\quad +\int_0^t\int_0^\infty m_Z(x,s,Z_5(s),\varphi(\hat{u}))\tilde{u}(x,s)|Y(s)|dxds.
\end{aligned}$$

Because $\tilde{\nu}$ is independent of n, letting $n \to \infty$, we further have

$$\begin{aligned}
&\int_0^\infty |v(x,t)|dx \\
&\quad \le \tilde{\nu} \int_0^t \int_0^\infty |v(x,s)|dxds \\
&\quad\quad - \int_0^t \int_0^\infty \gamma_Z(x,s,Z_4(s))\tilde{u}(x,s)|Y(s)|dxds \\
&\quad\quad + \int_0^t \int_0^\infty m_Z(x,s,Z_5(s),\varphi(\hat{u}))\tilde{u}(x,s)|Y(s)|dxds.
\end{aligned} \tag{2.2.14}$$

Substituting $(2.2.13)_2$ in (2.2.14) and applying Gronwall's inequality, we obtain that $v(x,t) = 0$ a.e. in D_T, which in turn implies that $Y(t) = 0$ in $(0,T)$. □

2.3 Monotone approximation and existence of the solution

We begin this section by constructing monotone sequences of upper and lower solutions. Suppose that $(\underline{u}^0(x,t), \underline{Z}^0(t))$ and $(\overline{u}^0(x,t), \overline{Z}^0(t))$ are a pair of lower and upper solutions of (2.1.1). We then set up two sequences $\{\underline{u}^k, \underline{Z}^k\}_{k=0}^\infty$ and $\{\overline{u}^k, \overline{Z}^k\}_{k=0}^\infty$ by the following procedure:

For $k = 1, 2, \ldots$, let $(\underline{u}^k, \underline{Z}^k)$ and $(\overline{u}^k, \overline{Z}^k)$ satisfy the system

$$\begin{aligned}
&\underline{u}_t^k + (g\underline{u}^k)_x + m(x,t,\overline{Z}^{k-1},\varphi(\overline{u}^{k-1}))\underline{u}^k \\
&\qquad\qquad = \mathcal{F}\underline{u}^{k-1} - \underline{u}^k \int_0^\infty \beta(x,y)\overline{u}^{k-1}(y,t)dy && \text{on } D_T, \\
&\frac{d\underline{Z}^k}{dt} = \left[f_1(t,\varphi(\underline{u}^{k-1})) - f_2(t,\overline{Z}^{k-1})\right]\underline{Z}^k && \text{on } (0,T), \\
&g(0,t)\underline{u}^k(0,t) = \int_0^\infty \gamma(y,t,\overline{Z}^{k-1}(t))\underline{u}^{k-1}(y,t)dy && \text{on } (0,T), \\
&\underline{u}^k(x,0) = u_0(x) && \text{in } [0,\infty), \\
&\underline{Z}^k(0) = Z_0
\end{aligned} \tag{2.3.1}$$

and

$$
\begin{aligned}
&\overline{u}^k_t + (g\overline{u}^k)_x + m(x,t,\underline{Z}^{k-1},\varphi(\underline{u}^{k-1}))\overline{u}^k \\
&\qquad\qquad\qquad = \mathcal{F}\overline{u}^{k-1} - \overline{u}^k\int_0^\infty \beta(x,y)\underline{u}^{k-1}(y,t)dy \qquad \text{on } D_T,\\
&\frac{d\overline{Z}^k}{dt} = \left[f_1(t,\varphi(\overline{u}^{k-1})) - f_2(t,\underline{Z}^{k-1})\right]\overline{Z}^k \qquad \text{on } (0,T),\\
&g(0,t)\overline{u}^k(0,t) = \int_0^\infty \gamma(y,t,\underline{Z}^{k-1}(t))\overline{u}^{k-1}(y,t)dy \qquad \text{on } (0,T),\\
&\overline{u}^k(x,0) = u_0(x) \qquad \text{in } [0,\infty),\\
&\overline{Z}^k(0) = Z_0.
\end{aligned}
\tag{2.3.2}
$$

The existence of solutions to problems (2.3.1) and (2.3.2) follows from the fact that (2.3.1) and (2.3.2) are both linear problems with local boundary conditions. We first show that $\underline{u}^0 \leq \underline{u}^1 \leq \overline{u}^1 \leq \overline{u}^0$ and $\underline{Z}^0 \leq \underline{Z}^1 \leq \overline{Z}^1 \leq \overline{Z}^0$. Let $v(x,t) = \underline{u}^0 - \underline{u}^1$ and $Y(t) = \underline{Z}^0 - \underline{Z}^1$. Then v and Y satisfy (2.2.11) with $A(x,t) = -\int_0^\infty \beta(x,y)\overline{u}^0(y,t)dy - m(x,t,\overline{Z}^0(t),\varphi(\overline{u}^0))$ and $\rho(t) = f_1(t,\varphi(\underline{u}^0)) - f_2(t,\overline{Z}^0(t))$. Thus by Remark 1, $v \leq 0$ and $Y \leq 0$, which imply $\underline{u}^0 \leq \underline{u}^1$ and $\underline{Z}^0 \leq \underline{Z}^1$. In a similar manner, it can be shown that $\overline{u}^1 \leq \overline{u}^0$ and $\overline{Z}^1 \leq \overline{Z}^0$.

We now claim that $(\underline{u}^1(x,t),\underline{Z}^1(t))$ and $(\overline{u}^1(x,t),\overline{Z}^1(t))$ are a lower solution and an upper solution of (2.1.1), respectively. Since $\underline{u}^0 \leq \underline{u}^1$, $\underline{Z}^0 \leq \underline{Z}^1$, $\overline{u}^1 \leq \overline{u}^0$ and $\overline{Z}^1 \leq \overline{Z}^0$, we have that from (2.3.1)

$$
\begin{aligned}
&\mathcal{F}\underline{u}^0 - \underline{u}^1\int_0^\infty \beta(x,y)\overline{u}^0(y,t)dy - m(x,t,\overline{Z}^0,\varphi(\overline{u}^0))\underline{u}^1\\
&\qquad \leq \mathcal{F}\underline{u}^1 - \int_0^\infty \beta(x,y)\underline{u}^1(x,t)\overline{u}^1(y,t)dy - m(x,t,\overline{Z}^1,\varphi(\overline{u}^1))\underline{u}^1,\\
&\left[f_1(t,\varphi(\underline{u}^0)) - f_2(t,\overline{Z}^0)\right]\underline{Z}^1 \leq \left[f_1(t,\varphi(\underline{u}^1)) - f_2(t,\overline{Z}^1)\right]\underline{Z}^1,\\
&\int_0^\infty \gamma(y,t,\overline{Z}^0(t))\underline{u}^0(y,t)dy \leq \int_0^\infty \gamma(y,t,\overline{Z}^1(t))\underline{u}^1(y,t)dy
\end{aligned}
$$

and from (2.3.2)

$$\begin{aligned}
&\mathcal{F}\overline{u}^0 - \overline{u}^1 \int_0^\infty \beta(x,y)\underline{u}^0(y,t)dy - m(x,t,\underline{Z}^0,\varphi(\underline{u}^0))\overline{u}^1 \\
&\qquad \geq \mathcal{F}\overline{u}^1 - \int_0^\infty \beta(x,y)\overline{u}^1(x,t)\underline{u}^1(y,t)dy - m(x,t,\underline{Z}^1,\varphi(\underline{u}^1))\overline{u}^1, \\
&\left[f_1(t,\varphi(\overline{u}^0)) - f_2(t,\underline{Z}^0)\right]\overline{Z}^1 \geq \left[f_1(t,\varphi(\overline{u}^1)) - f_2(t,\underline{Z}^1)\right]\overline{Z}^1, \\
&\int_0^\infty \gamma(y,t,\underline{Z}^0(t))\overline{u}^0(y,t)dy \geq \int_0^\infty \gamma(y,t,\underline{Z}^1(t))\overline{u}^1(y,t)dy.
\end{aligned}$$

The above inequalities imply that $(\underline{u}^1, \underline{Z}^1)$ and $(\overline{u}^1, \overline{Z}^1)$ satisfy (2.2.1) and (2.2.2), respectively, and hence $\underline{u}^1 \leq \overline{u}^1$ and $\underline{Z}^1 \leq \overline{Z}^1$.

We then assume that for some $k > 1$, $(\underline{u}^k, \underline{Z}^k)$ and $(\overline{u}^k, \overline{Z}^k)$ are a lower solution and an upper solution of (2.1.1), respectively. Arguing similarly, we can show that $\underline{u}^k \leq \underline{u}^{k+1} \leq \overline{u}^{k+1} \leq \overline{u}^k$, $\underline{Z}^k \leq \underline{Z}^{k+1} \leq \overline{Z}^{k+1} \leq \overline{Z}^k$ and that $(\underline{u}^{k+1}, \underline{Z}^{k+1})$ and $(\overline{u}^{k+1}, \overline{Z}^{k+1})$ are a lower solution and an upper solution of (2.1.1), respectively. Hence by induction, we obtain two monotone sequences $\{\underline{u}^k, \underline{Z}^k\}_{k=0}^\infty$ and $\{\overline{u}^k, \overline{Z}^k\}_{k=0}^\infty$ which satisfy

$$\underline{u}^0 \leq \underline{u}^1 \leq \cdots \leq \underline{u}^k \leq \overline{u}^k \leq \cdots \leq \overline{u}^1 \leq \overline{u}^0 \quad \text{a.e. in } \overline{D}_T,$$

$$\underline{Z}^0 \leq \underline{Z}^1 \leq \cdots \leq \underline{Z}^k \leq \overline{Z}^k \leq \cdots \leq \overline{Z}^1 \leq \overline{Z}^0 \quad \text{in } [0,T]$$

for each $k = 0, 1, 2, \ldots$. From the monotonicity of the sequences $\{\underline{u}^k, \underline{Z}^k\}_{k=0}^\infty$ and $\{\overline{u}^k, \overline{Z}^k\}_{k=0}^\infty$, it follows that there exist functions $(\underline{u}, \underline{Z})$ and $(\overline{u}, \overline{Z})$ such that $\underline{u}^k \to \underline{u}$, $\overline{u}^k \to \overline{u}$ in D_T, and $\underline{Z}^k \to \underline{Z}$, $\overline{Z}^k \to \overline{Z}$ in $(0,T)$. Clearly, $\underline{u} \leq \overline{u}$ a.e. in D_T and $\underline{Z} \leq \overline{Z}$ in $(0,T)$.

We now show that $\underline{u} = \overline{u}$ and $\underline{Z} = \overline{Z}$. To this end, let $v = \overline{u} - \underline{u}$ and $Y = \overline{Z} - \underline{Z}$. Since $\overline{u} \geq \underline{u}$ and $\overline{Z} \geq \underline{Z}$, $v(x,t) \geq 0$ and $Y(t) \geq 0$. In view of (2.2.3), by choosing $\xi(x,t) = \xi(x)$, where $\xi(x) \equiv 1$ for $0 \leq x \leq n$, $\xi(x) \equiv 0$ for $n+2 \leq x < \infty$, and $-1 \leq \xi' \leq 0$ for $n \leq x \leq n+2$, and $\eta(t) \equiv 1$, we have that

$$\int_0^n v(x,t)dx \le \int_0^t \int_0^\infty \gamma(x,s,\underline{Z}(s))v(x,s)dxds$$
$$+ \int_0^t \int_0^\infty [(\mathcal{F}\overline{u})(x,s) - (\mathcal{F}\underline{u})(x,s)]dxds$$
$$+ \int_0^t \int_0^\infty \underline{u}(x,s) \int_0^\infty \beta(x,y)v(y,s)dydxds$$
$$- \int_0^t \int_0^\infty \gamma_Z(x,s,Z_7(s))\underline{u}(x,s)Y(s)dxds$$
$$+ \int_0^t \int_0^\infty m_Z(x,s,Z_8(s),\varphi(\underline{u}))\underline{u}(x,s)Y(s)dxds$$
$$+ \int_0^t \int_0^\infty m_\varphi(x,s,\overline{Z}(s),\theta_5(s))\underline{u}(x,s) \int_0^\infty w(y)v(y,s)dydxds,$$

$$\begin{aligned} Y(t) \le & \int_0^t \underline{Z}(s) f_{1\varphi}(s,\theta_6(s)) \int_0^\infty w(x)v(x,s)dxds \\ & + \int_0^t \left[f_1(s,\varphi(\overline{u})) + f_{2Z}(s,Z_9(s))\underline{Z}(s)\right] Y(s)ds \end{aligned} \tag{2.3.3}$$

with Z_7, Z_8 and Z_9 between $\underline{Z}$ and $\overline{Z}$, θ_5 and θ_6 between $\varphi(\underline{u})$ and $\varphi(\overline{u})$. Since the right-hand side of the inequality $(2.3.3)_1$ is independent of n, the left-hand side can be replaced by $\int_0^\infty v(x,t)dx$. Then proceeding analogously as (2.2.9) and (2.2.10), we find that $v(x,t) = 0$ and $Y(t) = 0$, i.e., $\underline{u} = \overline{u}$ and $\underline{Z} = \overline{Z}$. Defining this common limit by (u, Z), we see that (u, Z) is the solution of (2.1.1). In summary, we have the following existence-uniqueness result.

Theorem 2.3.1. Suppose that hypotheses (H1)-(H8) hold. Furthermore, suppose that $(\underline{u}^0, \underline{Z}^0)$ and $(\overline{u}^0, \overline{Z}^0)$ are a nonnegative lower solution and a nonnegative upper solution of (2.1.1), respectively. Then there exist monotone sequences $\{\underline{u}^k, \underline{Z}^k\}_{k=0}^\infty$ and $\{\overline{u}^k, \overline{Z}^k\}_{k=0}^\infty$ converging to the unique solution (u, Z) of (2.1.1).

Remark 2. As an example, for a large class of initial data of u such as $u_0(x) = O(e^{-x})$ as $x \to \infty$, we can construct a pair of nonnegative lower and upper solutions of (2.1.1) as follows: Let $\underline{u}^0(x,t) = 0$, $\overline{u}^0(x,t) = ce^{bt}/(1 + a^2x^2)$, $\underline{Z}^0(t) = 0$, and $\overline{Z}^0(t) = Z_0 e^{dt}$, where

a, b, c, and d are positive constants to be determined. First choose a so large that

$$a \geq \pi \|\gamma\|_\infty / 2 \min_{[0,1]} g(0,t).$$

Fix this a and choose c large enough such that $c/(1+a^2x^2) \geq u_0(x)$ for $0 \leq x < \infty$. We then determine b. Through a routine calculation, we find

$$\begin{aligned}
&\int_0^x \frac{dy}{[1+a^2(x-y)^2](1+a^2y^2)} \\
&= \frac{2}{a^2x}\left[\frac{ax\tan^{-1}(ax)+\log(1+a^2x^2)}{4+a^2x^2}\right] \\
&\leq \frac{2(1+\pi)}{a(1+a^2x^2)}.
\end{aligned}$$

Thus we can choose b sufficiently large such that

$$b \geq \frac{3c}{a}(1+\pi)\|\beta\|_\infty + 2\max_{\overline{D}_1} g(x,t) + \max_{\overline{D}_1} |g_x(x,t)|.$$

Finally, we can choose d large enough such that $d \geq \|f_1\|_\infty$. Clearly, $(\underline{u}^0, \underline{Z}^0)$ is a lower solution. And it is easy to verify that the above constructed $(\overline{u}^0, \overline{Z}^0)$ is a desired upper solution of (2.1.1) on D_T with $T = \min\{1, 1/b\}$.

We now show that the solution of (2.1.1) possesses the following property.

Theorem 2.3.2. Suppose that hypotheses (H1)-(H8) hold. Then for the nonnegative solution $(u(x,t), Z(t))$ of (2.1.1), $P(t) = \int_0^\infty u(x,t)dx$ and $Z(t)$ are continuous in the existence interval.

Proof. To show $P \in C[0,T]$, by the continuity assumptions on the parameters, it suffices to establish the following equality:

$$\begin{aligned}
\int_0^\infty & u(x,t)dx \\
&= \int_0^\infty u(x,0)dx + \int_0^t \int_0^\infty (\mathcal{F}u)(x,s)dxds \\
&\quad + \int_0^t \int_0^\infty \gamma(x,s,Z(s))u(x,s)dxds \\
&\quad - \int_0^t \int_0^\infty \int_0^\infty \beta(x,y)u(x,s)u(y,s)dydxds \\
&\quad - \int_0^t \int_0^\infty m(x,s,Z(s),\varphi(u))u(x,s)dxds.
\end{aligned} \tag{2.3.4}$$

To this end, we again choose $\xi(x,t) = \xi(x)$, where $\xi(x) \equiv 1$ for $0 \le x \le n$, $\xi(x) \equiv 0$ for $n+2 \le x < \infty$, and $-1 \le \xi' \le 0$ for $n \le x \le n+2$. By the definition of the solution of (2.1.1), we find

$$\begin{aligned}
&\Bigg| \int_0^\infty u(x,t)dx - \int_0^\infty u(x,0)dx - \int_0^t \int_0^\infty (\mathcal{F}u)(x,s)dxds \\
&\quad - \int_0^t \int_0^\infty \gamma(x,s,Z(s))u(x,s)dxds + \int_0^t \int_0^\infty \int_0^\infty \beta(x,y)u(x,s)u(y,s)dydxds \\
&\quad + \int_0^t \int_0^\infty m(x,s,Z(s),\varphi(u))u(x,s)dxds \Bigg| \\
&= \Bigg| \int_n^\infty [u(x,t)-u(x,0)][1-\xi(x)]dx + \int_0^t \int_n^\infty g(x,s)\xi'(x)u(x,s)dxds \\
&\qquad - \int_0^t \int_n^\infty (\mathcal{F}u)(x,s)[1-\xi(x)]dxds \\
&\qquad + \int_0^t \int_n^\infty u(x,s)[1-\xi(x)] \int_0^\infty \beta(x,y)u(y,s)dydxds \\
&\qquad + \int_0^t \int_n^\infty m(x,s,Z(s),\varphi(u))[1-\xi(x)]u(x,s)dxds \Bigg| \\
&\le \left(2 + \|g\|_\infty T + \frac{3}{2}\|\beta\|_\infty T \sup_{[0,T]} \|u(\cdot,t)\|_1 + \|m\|_\infty T \right) \sup_{[0,T]} \int_n^\infty u(x,t)dx.
\end{aligned}$$

Since $u \in L^\infty((0,T); L^1(0,\infty) \cap L^\infty(0,\infty))$, $\sup_{[0,T]} \int_n^\infty u(x,t)dx \to 0$ as $n \to \infty$, which leads to (2.3.4).

On the other hand, since f_1 and f_2 are continuous, it follows from $(2.1.2)_2$ that $Z \in C[0,T]$. □

Bibliography

[1] A.S. Ackleh, Parameter estimation in a structured algal coagulation fragmentation model, *Nonlinear Anal. TMA* **28** (1997), 837-854.

[2] A.S. Ackleh and K. Deng, On a first order hyperbolic coagulation model, *Math. Methods Appl. Sci.* **26** (2003), 703–715.

[3] A.S. Ackleh and R.R. Ferdinand, A nonlinear phytoplankton aggregation model with light shading, *SIAM J. Appl. Math.* **60** (1999), 316-336.

[4] A.S. Ackleh and B.G. Fitzpatrick, Modeling aggregation and growth processes in an algal population model: analysis and computions, *J. Math. Biol.* **35** (1997), 480-502.

[5] A.S. Ackleh, T.G. Hallam, and H.C. Muller-Landau, Estimation of sticking and contact efficiencies in aggregation of phytoplankton: The 1993 SIGMA tank experiment, *Deep-Sea Res.* **42** (1995), 185-201.

[6] A.S. Ackleh, T.G. Hallam, and W.O. Smith, Influences of aggregation and grazing on phytoplankton dynamics adn fluxes: an individual-based modeling approach, *Nonlinear World* **1** (1994), 473-492.

[7] O. Diekmann, M. Gyllenberg, and H. Thieme, Perturbing evolutionary systems by step responses and cumulative outputs, *Differential Integral Equations* **8** (1995), 1205-1244.

[8] D.J. McLaughlin, W. Lamb, and A.C. McBride, A semigroup approach to fragmentation models, *SIAM J. Math. Anal.* **28** (1997), 1158–1172.

[9] D.J. McLaughlin, W. Lamb, and A.C. McBride, An existence and uniqueness result for a coagulation and multiple-fragmentation equation, *SIAM J. Math. Anal.* **28** (1997), 1173–1190.

[10] D.J. McLaughlin, W. Lamb, and A.C. McBride, Existence and uniqueness results for the non-autonomous coagulation and multiple-fragmentation equation, *Math. Methods Appl. Sci.* **21** (1998), 1067–1084.

Chapter 3

The Asymptotic Behavior of Solutions to a Coupled System of Nonlinear Size-Structured Populations

We study a size-structured model which describes the dynamics of n-subpopulations with nonlinear growth, reproduction and mortality rates. We establish existence and uniqueness results for the solutions. We also show that there exists a compact global attractor for the trajectories of the dynamical system defined by the solutions of this model. In addition, we consider two open reproduction cases and study the asymptotic dynamics for these special cases.

3.1 Introduction

We study the following initial-boundary value problem that describes the dynamics of coupled size-structured subpopulations with nonlinear growth, mortality and reproduction depending on the individual's size and the total population:

$$\begin{aligned}
&(u_i)_t + (g_i(x, P(t))u_i)_x + m_i(x, P(t))u_i = 0 && 0 < x \leq x_{\max} \leq \infty, \quad t > 0,\\
&g_i(0, P(t))u_i(0, t) = c_i(t) + \sum_{j=1}^{n} \int_0^{x_{\max}} \gamma_{i,j}\beta_j(x, P(t))u_j(x, t)dx && t > 0,\\
&u_i(x, 0) = u_{i0}(x) && 0 \leq x \leq x_{\max} \leq \infty.
\end{aligned} \tag{3.1.1}$$

Here $u_i(x,t),\ i=1,2,\ldots,n$, is the density of individuals in the ith subpopulation having size x at time t. The function $P(t)=\sum_{i=1}^{n}\int_0^{x_{\max}} u_i(x,t)dx$ represents the total population at time t. The functions g_i, m_i and β_i denote respectively the growth rate, mortality rate and reproduction rate of an individual in the ith subpopulation. The constant parameter $0 \le \gamma_{i,j} \le 1$ represents the probability that an individual of the jth subpopulation will reproduce an individual of the ith subpopulation. Clearly, $\sum_{j=1}^{n}\gamma_{i,j} = \sum_{i=1}^{n}\gamma_{i,j} = 1,\ i,j=1,2,\ldots,n$. The function c_i is the inflow rate of the ith subpopulation of zero-size individuals from an external source (e.g., seeds flown by wind).

In [1], an implicit finite difference approximation was developed to obtain the existence-uniqueness of weak solutions as well as convergence of the difference approximations for the quasilinear problem with $x_{\max} < \infty$. In [5], the contraction mapping argument was used to obtain the existence-uniqueness of solutions when $x_{\max} \le \infty$ and with the vital rates of each subpopulation depending only on the total population due to competition.

The goal of this chapter is twofold. On the one hand, we will generalize existence and uniqueness results for problem (3.1.1) using the contraction mapping argument. On the other hand, we will study the asymptotic behavior of solutions to the model using asymptotic theory of dissipative systems.

The chapter is organized as follows. In section 3.2, we establish existence and uniqueness results for problem (3.1.1). In section 3.3, we show the continuous dependence on initial conditions. In section 3.4, we show that the solutions of problem (3.1.1) generate a C_0-semigroup and that there exists a compact global attractor for the trajectories of the dynamical system defined by the solutions of problem (3.1.1). In section 3.5, we present two open reproduction cases and analyze the asymptotic behavior for these cases. For completeness, we give the proof of local existence and uniqueness results for problem (3.1.1) in the Appendix.

3.2 Existence and uniqueness results

For simplicity, let $\Omega := [0, x_{\max}) \times [0,\infty)$. In order to carry out our program, we impose the following assumptions on the parameters in problem (3.1.1):

(H1) $g_i(x,P)$ is a strictly positive Lipschitz function on Ω with constant L_{g_i} and continuously differentiable with respect to x on Ω. In addition, $\lim_{x\to x_{\max}} g_i(x,P) = 0$ for

$P \in [0, \infty)$ and $g_i(x, P)$ is uniformly bounded on Ω with $0 \le g_i \le g_M$.

(H2) $m_i(x, P)$ is a nonnegative Lipschitz function on Ω with constant L_{m_i}.

(H3) $c_i(t)$ is a nonnegative continuous function and uniformly bounded for $0 \le t < \infty$ with $0 \le c_i \le c_M$.

(H4) $\beta_i(x, P)$ is a nonnegative Lipschitz function on Ω with constant L_{β_i} and uniformly bounded on Ω with $0 \le \beta_i \le \beta_M$.

(H5) $u_{i0} \in L^1(0, x_{\max})$ and $u_{i0} \ge 0$.

We begin with the definition of the solution to problem (3.1.1).

Definition 3.2.1. A nonnegative function $u(x, t) = (u_1(x, t), u_2(x, t), \ldots, u_n(x, t))$ on $[0, x_{\max}) \times [0, T)$, with $u(\cdot, t)$ integrable is a solution of (3.1.1) if $P(t) = \displaystyle\sum_{i=1}^{n} \int_0^{x_{\max}} u_i(x, t)dx$ is a continuous function on $[0, T)$ and for $i = 1, 2, \ldots, n$, $u_i(x, t)$ satisfies $(3.1.1)_2$, $(3.1.1)_3$, and the equation

$$Du_i(x, t) = -\tilde{m}_i(x, P(t))u_i(x, t) \qquad 0 < x < x_{\max},\ 0 < t < T \tag{3.2.1}$$

with

$$Du_i(x, t) = \lim_{h \to 0} \frac{u_i(X_i(t + h; x, t), t + h) - u_i(x, t)}{h}, \tag{3.2.2}$$

where $\tilde{m}_i(x, P(t)) := m_i(x, P(t)) + (g_i)_x(x, P(t))$ and $X_i(t; x_0, t_0)$ is the solution of the equation for the characteristic curves given by

$$\begin{cases} \dfrac{d}{dt}x(t) = g_i(x(t), P(t)) \\ x(t_0) = x_0. \end{cases} \tag{3.2.3}$$

From (H1) we know that the function X_i is strictly increasing. Thus a unique inverse function $\tau_i(x; x_0, t_0)$ exists. Let $z_i(t) = X_i(t; 0, 0)$, the characteristic through the origin, and $B_i(t) = c_i(t) + \displaystyle\sum_{j=1}^{n} \int_0^{x_{\max}} \gamma_{i,j}\beta_j(x, P(t))u_j(x, t)dx$, the inflow of newborns in the ith subpopulation at time t. In the following, we reduce problem (3.1.1) to a system of coupled equations for $P(t)$ and $B_i(t)$ by using the method of characteristics.

Integrating (3.2.1) along the characteristics, we have

$$u_i(x,t) = \frac{B_i(\tau_i(0;x,t))}{g_i(0,P(\tau_i(0;x,t)))} \exp\left(-\int_{\tau_i(0;x,t)}^t \tilde{m}_i(X_i(s;0,\tau_i(0;x,t)),P(s))ds\right) \quad x < z_i(t),$$
$$u_i(x,t) = u_{i0}(X_i(0;x,t)) \exp\left(-\int_0^t \tilde{m}_i(X_i(s;x,t),P(s))ds\right) \quad x \geq z_i(t). \tag{3.2.4}$$

Then integrating (3.2.4) with respect to x and summing over the indices $i = 1, 2, \ldots, n$, we obtain an integral equation for $P(t)$,

$$\begin{aligned} P(t) &= \sum_{i=1}^n \left[\int_0^{z_i(t)} \frac{B_i(\tau_i(0;x,t))}{g_i(0,P(\tau_i(0;x,t)))} \exp\left(-\int_{\tau_i(0;x,t)}^t \tilde{m}_i(X_i(s;0,\tau_i),P(s))ds\right)dx \right.\\ &\qquad \left. + \int_{z_i(t)}^{x_{\max}} u_{i0}(X_i(0;x,t)) \exp\left(-\int_0^t \tilde{m}_i(X_i(s;x,t),P(s))ds\right)dx\right] \\ &= \sum_{i=1}^n \left[\int_0^t B_i(\eta) e^{-\int_\eta^t m_i(X_i(s;0,\eta),P(s))ds} d\eta + \int_0^{x_{\max}} u_{i0}(\xi) e^{-\int_0^t m_i(X_i(s;\xi,0),P(s))ds} d\xi\right]. \end{aligned} \tag{3.2.5}$$

Then substituting (3.2.4) in the definition of $B_i(t)$ and using the same changes of variable as those used in (3.2.5), we find an integral equation for $B_i(t)$,

$$\begin{aligned} B_i(t) = c_i(t) &+ \sum_{j=1}^n \left[\int_0^t \gamma_{i,j}\beta_j(X_j(t;0,\eta),P(t))B_j(\eta) e^{-\int_\eta^t m_j(X_j(s;0,\eta),P(s))ds} d\eta \right.\\ &\left. + \int_0^{x_{\max}} \gamma_{i,j}\beta_j(X_j(t;\xi,0),P(t))u_{j0}(\xi) e^{-\int_0^t m_j(X_j(s;\xi,0),P(s))ds} d\xi\right]. \end{aligned} \tag{3.2.6}$$

On the one hand, if $P(t)$ and $B_i(t)$ are nonnegative continuous solutions of (3.2.5)-(3.2.6), then $u(x,t)$ defined by (3.2.4) is a solution of (3.1.1). On the other hand, if $u(x,t)$ is a solution of (3.1.1), then $P(t)$ and $B_i(t)$ are nonnegative continuous solutions of (3.2.5)-(3.2.6). Therefore, in order to obtain the existence and uniqueness results for problem (3.1.1), we only need to study the solvability of the system of integral equations (3.2.5)-(3.2.6). To this end, for $K > ||u_0||_{L^1} = \sum_{i=1}^n \int_0^{x_{\max}} u_{i0}(x)dx$, let $S_{T,K} = \{f(t) \in C[0,T] : f(0) = ||u_0||_{L^1},\ 0 \leq f(t) \leq K\}$. For each $P \in S_{T,K}$, let $B_i(t) \in C[0,T]$ be the unique nonnegative solution of (3.2.6), and we define the operator $\mathcal{P} : S_{T,K} \to C[0,T]$ in such a way that $\mathcal{P}(P)(t)$ is the right hand side of (3.2.5) for these $P(t)$ and $B_i(t)$.

As can be seen in the Appendix, the local existence and uniqueness results (Lemma 3.2.2 and Theorem 3.2.3) for problem (3.1.1) can be established using similar techniques as those in [5] and [11].

Lemma 3.2.2. Suppose that hypotheses (H1)-(H5) hold. Then there exists a value $T > 0$ for which $\mathcal{P}$ has a unique fixed point.

Theorem 3.2.3. Suppose that hypotheses (H1)-(H5) hold. Then there exists a value $T > 0$ such that problem (3.1.1) has a unique solution up to time T.

In order to establish the global existence-uniqueness result for problem (3.1.1), we introduce the following upper bound on $P(t)$ for $t \in [0, T]$.

Lemma 3.2.4. Let $u(x,t)$ be a solution of (3.1.1) up to time T. Then $P(t)$ satisfies the following bound

$$P(t) \leq \left(\|u_0\|_{L^1} + \frac{nc_M}{\beta_M} \right) e^{\beta_M t} - \frac{nc_M}{\beta_M} \quad \text{for} \quad t \in [0, T]. \tag{3.2.7}$$

Proof. Let $P_i(t) = \int_0^{x_{\max}} u_i(x,t)dx$, then $P(t) = \sum_{i=1}^{n} P_i(t)$. Integrating (3.2.4) with respect to x, we obtain an integral equation for $P_i(t)$, $i = 1, 2, \ldots, n$,

$$P_i(t) = \int_0^t B_i(\eta) e^{-\int_\eta^t m_i(X_i(s;0,\eta),P(s))ds} d\eta + \int_0^{x_{\max}} u_{i0}(\xi) e^{-\int_0^t m_i(X_i(s;\xi,0),P(s))ds} d\xi. \tag{3.2.8}$$

Differentiating (3.2.8) with respect to t, we find

$$P_i'(t) = c_i(t) + \sum_{j=1}^{n} \int_0^{x_{\max}} \gamma_{i,j} \beta_j(x,P) u_j(x,t)dx - \int_0^{x_{\max}} m_i(x,P) u_i(x,t)dx. \tag{3.2.9}$$

Thus we have

$$\begin{aligned} P'(t) &= \sum_{i=1}^{n} c_i(t) + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \int_0^{x_{\max}} \gamma_{i,j} \beta_j(x,P) u_j(x,t)dx - \int_0^{x_{\max}} m_i(x,P) u_i(x,t)dx \right) \\ &= \sum_{i=1}^{n} c_i(t) + \sum_{i=1}^{n} \int_0^{x_{\max}} \left(\beta_i(x,P) - m_i(x,P)\right) u_i(x,t)dx \\ &\leq nc_M + \beta_M P(t), \end{aligned} \tag{3.2.10}$$

which implies (3.2.7). □

Proceeding analogously as in the proof of Theorem 3 in [11], we have the following global existence-uniqueness result.

Theorem 3.2.5. Suppose that hypotheses (H1)-(H5) hold. Then problem (3.1.1) has a unique solution for all positive time.

3.3 Continuous dependence on initial conditions

In this section, we establish the continuous dependence on initial conditions. First, we show that the fixed point of the operator $\mathcal{P}$ associated with an initial condition depends continuously on this initial condition.

Lemma 3.3.1. Let $P_u^*(t)$ and $P_v^*(t)$ be the fixed points associated with the initial conditions u_0 and v_0, respectively. Then

$$|P_u^*(t) - P_v^*(t)| \leq \frac{nt\beta_M e^{n\beta_M t} + 1}{1-L}\|u_0 - v_0\|_{L^1}, \tag{3.3.1}$$

where L is the contraction constant of the operator $\mathcal{P}$.

Proof. From (3.2.5), we have

$$P_u^*(t) = \sum_{i=1}^n \left[\int_0^t B_{ui}(\eta)e^{-\int_\eta^t m_i(X_{ui}(s;0,\eta),P_u^*(s))ds}d\eta + \int_0^{x_{\max}} u_{i0}(\xi)e^{-\int_0^t m_i(X_{ui}(s;\xi,0),P_u^*(s))ds}d\xi\right]$$

and

$$P_v^*(t) = \sum_{i=1}^n \left[\int_0^t B_{vi}(\eta)e^{-\int_\eta^t m_i(X_{vi}(s;0,\eta),P_v^*(s))ds}d\eta + \int_0^{x_{\max}} v_{i0}(\xi)e^{-\int_0^t m_i(X_{vi}(s;\xi,0),P_v^*(s))ds}d\xi\right],$$

where X_{ui} and X_{vi} are the characteristics corresponding to u_0 and v_0, respectively, and $B_{ui}(\eta)$ and $B_{vi}(\eta)$ satisfy

$$\begin{aligned} B_{ui}(t) = c_i(t) + \sum_{j=1}^n \Bigg[&\int_0^t \gamma_{i,j}\beta_j(X_{uj}(t;0,\eta),P_u^*(t))B_{uj}(\eta)e^{-\int_\eta^t m_j(X_{uj}(s;0,\eta),P_u^*(s))ds}d\eta \\ &+ \int_0^{x_{\max}} \gamma_{i,j}\beta_j(X_{uj}(t;\xi,0),P_u^*(t))u_{j0}(\xi)e^{-\int_0^t m_j(X_{uj}(s;\xi,0),P_u^*(s))ds}d\xi\Bigg] \end{aligned} \tag{3.3.2}$$

and

$$\begin{aligned} B_{vi}(t) = c_i(t) + \sum_{j=1}^n \Bigg[&\int_0^t \gamma_{i,j}\beta_j(X_{vj}(t;0,\eta),P_v^*(t))B_{vj}(\eta)e^{-\int_\eta^t m_j(X_{vj}(s;0,\eta),P_v^*(s))ds}d\eta \\ &+ \int_0^{x_{\max}} \gamma_{i,j}\beta_j(X_{vj}(t;\xi,0),P_v^*(t))v_{j0}(\xi)e^{-\int_0^t m_j(X_{vj}(s;\xi,0),P_v^*(s))ds}d\xi\Bigg], \end{aligned} \tag{3.3.3}$$

respectively.

Let

$$P_w^*(t) = \sum_{i=1}^n \left[\int_0^t B_{wi}(\eta)e^{-\int_\eta^t m_i(X_{vi}(s;0,\eta),P_v^*(s))ds}d\eta + \int_0^{x_{\max}} u_{i0}(\xi)e^{-\int_0^t m_i(X_{vi}(s;\xi,0),P_v^*(s))ds}d\xi\right]$$

with $B_{wi}(\eta)$ satisfying

$$B_{wi}(t) = c_i(t) + \sum_{j=1}^{n} \left[\int_0^t \gamma_{i,j}\beta_j(X_{vj}(t;0,\eta), P_v^*(t))B_{wj}(\eta)e^{-\int_\eta^t m_j(X_{vj}(s;0,\eta),P_v^*(s))ds}d\eta \right.$$
$$\left. + \int_0^{x_{\max}} \gamma_{i,j}\beta_j(X_{vj}(t;\xi,0), P_v^*(t))u_{j0}(\xi)e^{-\int_0^t m_j(X_{vj}(s;\xi,0),P_v^*(s))ds}d\xi \right]. \tag{3.3.4}$$

Using the contraction mapping argument for $P_u^*(t)$ and $P_w^*(t)$, we have

$$|P_u^*(t) - P_w^*(t)| \le L|P_u^*(t) - P_v^*(t)|.$$

Since

$$|B_{vi}(t) - B_{wi}(t)| \le \sum_{j=1}^{n} \left[\int_0^t \beta_M |B_{vj}(\eta) - B_{wj}(\eta)|d\eta + \beta_M \int_0^{x_{\max}} |u_{j0}(\xi) - v_{j0}(\xi)|d\xi \right],$$

it follows that

$$\sum_{i=1}^{n} |B_{vi}(t) - B_{wi}(t)| \le n\beta_M \int_0^t \sum_{j=1}^{n} |B_{vj}(\eta) - B_{wj}(\eta)|d\eta + n\beta_M \|u_0 - v_0\|_{L^1},$$

which by Gronwall's inequality leads to

$$\sum_{i=1}^{n} |B_{vi}(t) - B_{wi}(t)| \le n\beta_M e^{n\beta_M t} \|u_0 - v_0\|_{L^1}.$$

Thus, we have

$$|P_v^*(t) - P_w^*(t)| \le nt\beta_M e^{n\beta_M t} \|u_0 - v_0\|_{L^1} + \|u_0 - v_0\|_{L^1}.$$

We then obtain

$$\begin{aligned} |P_u^*(t) - P_v^*(t)| &\le |P_u^*(t) - P_w^*(t)| + |P_v^*(t) - P_w^*(t)| \\ &\le L|P_u^*(t) - P_v^*(t)| + \left(1 + nt\beta_M e^{n\beta_M t}\right) \|u_0 - v_0\|_{L^1}. \end{aligned}$$

Therefore, we find

$$|P_u^*(t) - P_v^*(t)| \le \frac{nt\beta_M e^{n\beta_M t} + 1}{1 - L} \|u_0 - v_0\|_{L^1}.$$

□

By means of (3.3.1), we now establish the continuous dependence on initial conditions.

Theorem 3.3.2. Let $u_k(x,t) = (u_{k1}(x,t), u_{k2}(x,t), \ldots, u_{kn}(x,t)), k = 1,2,$ be the solution of problem (3.1.1) in the sense of definition 3.2.1 corresponding to the initial condition $u_{k0} = (u_{k10}, u_{k20}, \ldots, u_{kn0})$. Then, for fixed $t > 0$, integrable initial condition u_{10}, any $\epsilon > 0$ and $1 \le i \le n$, there exists $\delta = \delta(\epsilon, u_{10}, t) > 0$ such that if $\|u_{10} - u_{20}\|_{L^1} < \delta$ then $\|u_{1i}(\cdot,t) - u_{2i}(\cdot,t)\|_{L^1} < \epsilon$.

Proof. For $1 \le i \le n$, let X_{ki} $(k = 1,2)$ be the solution of the initial value problem $\alpha'(s) = g_i(\alpha, P^k(s))$ with $\alpha(t) = x$ and $P^k(t) = \sum_{i=1}^{n} \int_0^{x_{\max}} u_{ki}(x,t)dx$. Let τ_{ki} and ξ_{ki} be given by $X_{ki}(\tau_{ki}; x,t) = 0$ and $X_{ki}(0;x,t) = \xi_{ki}$, respectively. In addition, let $z_{ki}(t) = X_{ki}(t;0,0)$. For simplicity, let $\pi_{ki}(\tau_{ki}) = e^{-\int_{\tau_{ki}}^t \tilde{m}_i(X_{ki}(s;0,\tau_{ki}),P^k(s))ds}$ and $\tilde{\pi}_{ki} = e^{-\int_0^t \tilde{m}_i(X_{ki}(s;x,t),P^k(s))ds}$.

For each initial condition, studying the solutions of problem (3.1.1) through (x,t) and estimating their difference at any time $s \le t$, we have

$$\begin{aligned} |X_{1i}(s;x,t) - X_{2i}(s;x,t)| &\le L_{g_i} \int_s^t \left|P^1(\tau) - P^2(\tau)\right| e^{L_{g_i}(\tau - s)} d\tau \\ &\le L_{g_i} t \left(\frac{nt\beta_M e^{n\beta_M t} + 1}{1 - L} \right) e^{L_{g_i}(t-s)} \|u_{10} - u_{20}\|_{L^1}. \end{aligned} \tag{3.3.5}$$

Thus, for all $s \le t$, the distance between characteristics is uniformly bounded for all x and tends uniformly to zero as $\|u_{10} - u_{20}\|_{L^1}$ tends to zero.

Now, we assume $z_{1i}(t) \le z_{2i}(t)$. Then it follows from (3.2.4) that

$$\begin{aligned} &\int_0^{x_{\max}} |u_{1i}(x,t) - u_{2i}(x,t)|\, dx \\ &\quad \le \int_0^{z_{1i}(t)} \left| \frac{B_{1i}(\tau_{1i})}{g_i(0, P^1(\tau_{1i}))} - \frac{B_{2i}(\tau_{2i})}{g_i(0, P^2(\tau_{2i}))} \right| \pi_{1i}(\tau_{1i}) dx \\ &\quad + \int_0^{z_{1i}(t)} \frac{B_{2i}(\tau_{2i})}{g(0, P^2(\tau_{2i}))} |\pi_{1i}(\tau_{1i}) - \pi_{2i}(\tau_{2i})|\, dx \\ &\quad + \int_{z_{1i}(t)}^{z_{2i}(t)} \left| u_{1i0}(\xi_{1i})\tilde{\pi}_{1i} - \frac{B_{2i}(\tau_{2i})}{g_i(0, P^2(\tau_{2i}))} \pi_{2i}(\tau_{2i}) \right| dx \\ &\quad + \int_{z_{2i}(t)}^{x_{\max}} u_{1i0}(\xi_{1i})\, |\tilde{\pi}_{1i} - \tilde{\pi}_{2i}|\, dx \\ &\quad + \int_{z_{2i}(t)}^{x_{\max}} |u_{1i0}(\xi_{1i}) - u_{1i0}(\xi_{2i})|\, \tilde{\pi}_{2i} dx \\ &\quad + \int_{z_{2i}(t)}^{x_{\max}} |u_{1i0}(\xi_{2i}) - u_{2i0}(\xi_{2i})|\, \tilde{\pi}_{2i} dx. \end{aligned} \tag{3.3.6}$$

Thus, the proof can be completed by using similar arguments as those of Theorem 2 in [11]. □

3.4 Existence of a compact global attractor

In this section, we establish the existence of a compact global attractor for the trajectories of the dynamical system defined by the solutions of (3.1.1).

Let $X = \{u = (u_1, u_2, \ldots, u_n) : u_i(x) \geq 0 \ a.e. \text{ and } \ u_i(x) \in L^1(0, x_{\max}), i = 1, 2, \ldots, n\}$ and the family of maps $\{S(t) : X \to X, \ t \in [0, \infty)\}$ defined by $S(t)u_0 = u(x,t)$, where $u_0 = (u_{10}, u_{20}, \ldots, u_{n0})$ and $u = (u_1, u_2, \ldots, u_n)$ are the initial condition and the solution of (3.1.1) corresponding to this initial condition, respectively.

The following results (Lemmas 3.4.1—3.4.3) will be used in the sequel. They have been established in [15] and [17]. For details, see [13], [15] and [17].

Lemma 3.4.1. Let $S(t)$ be a C_0 semigroup such that $S(t) = U(t) + W(t)$. If $U(t)$ is compact and there exists a continuous function $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $k(t,r) \to 0$ when $t \to \infty$ and $\|W(t)\Phi\| \leq k(t,r)$ if $\|\Phi\| \leq r$, then $S(t)$ is asymptotically smooth.

Lemma 3.4.2. A closed and bounded set $\mathcal{B}$ of $L_1(0, x_{\max})$ is compact if and only if the following conditions are satisfied:

a) $\lim\limits_{h\to 0} \int_0^{x_{\max}} |\Phi(x+h) - \Phi(x)|\, dx = 0$ uniformly for $\Phi \in \mathcal{B}$ with $\Phi(x) = 0$ if $x \notin (0, x_{\max})$.

b) $\lim\limits_{h\to\infty} \int_h^{\infty} |\Phi(x)|\, dx = 0$ uniformly for $\Phi \in \mathcal{B}$ (if $x_{\max} = \infty$).

Lemma 3.4.3. An asymptotically smooth C_0-semigroup defined in a complete metric space that is point dissipative and for which orbits of bounded sets are bounded has a compact global attractor.

Using the above results, we now establish the existence of a compact global attractor for the trajectories of the dynamical system defined by the solutions of (3.1.1).

First, we have $S(0)u_0 = u(x,0) = u_0$ and the existence-uniqueness of the solution of (3.1.1) in the sense of Definition 3.2.1 implies the semigroup property $S(t)u_0 = S(t-\tau)S(\tau)u_0$. Furthermore, using the similar argument as that of continuous dependence on initial conditions, we can establish the continuous dependence on time. Thus, $S(t)$ is a C_0-semigroup on X.

Next, we show that $S(t)$ is asymptotically smooth and point dissipative. To this end, let $S(t) = U(t)+W(t)$ such that $U(t)\Phi = ((U(t)\Phi)_1, (U(t)\Phi)_2, \ldots, (U(t)\Phi)_i, \ldots, (U(t)\Phi)_n)$ and $W(t)\Phi = ((W(t)\Phi)_1, (W(t)\Phi)_2, \ldots, (W(t)\Phi)_i, \ldots, (W(t)\Phi)_n)$ with

$$(U(t)\Phi)_i = \begin{cases} u_i(x,t) & \text{a.e. } \ x \le z_i(t) \\ 0 & \text{a.e. } \ x > z_i(t) \end{cases} \tag{3.4.1}$$

and

$$(W(t)\Phi)_i = \begin{cases} 0 & \text{a.e. } \ x \le z_i(t) \\ u_i(x,t) & \text{a.e. } \ x > z_i(t) \end{cases} \tag{3.4.2}$$

for all initial distribution $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_i, \ldots, \Phi_n) \in X$. Here $z_i(t) = X_i(t; 0, 0)$ is the characteristic through the origin as before, and below we use the notation $z_{i\Phi}(t)$ to indicate that $z_i(t)$ depends on the initial distribution Φ.

Lemma 3.4.4. Suppose that hypotheses (H1)-(H5) hold; then $U(t)$ is compact.

Proof. To show $U(t)$ is compact, it suffices to show that $\overline{U(t)B}$ is compact for every bounded set $B \subset X$ from a certain t onward. Since B is a bounded set and $P(t)$ satisfies (3.2.7), $U(t)B$ is bounded and therefore $\overline{U(t)B}$ is closed and bounded in X for any t. Noting that $(U(t)\Phi)_i(x) = 0$ when $x \ge h > tg_M$ for $i = 1, 2, \ldots, n$, to show $\overline{U(t)B}$ is compact when $x_{\max} = \infty$, we only need to show

$$\lim_{h\to 0} \int_0^\infty |(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)|\, dx = 0 \tag{3.4.3}$$

uniformly for $\Phi \in B$.

First, we consider the case when $x_{\max} = \infty$.

Depending on the sign of h, (3.4.3) becomes

$$\lim_{h\to 0^+} \sigma(h) := \lim_{h\to 0^+} \int_0^{z_{i\Phi}(t)} |(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)| dx = 0$$

and

$$\lim_{h\to 0^-} \sigma(h) := \lim_{h\to 0^-} \int_0^{z_{i\Phi}(t)-h} |(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)| dx = 0,$$

where $\sigma(h)$ can be written as follows:

(a) If $0 < h < z_{i\Phi}(t)$, then

$$\sigma(h) = \int_0^{z_{i\Phi}(t)-h} |(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)| dx + \int_{z_{i\Phi}(t)-h}^{z_{i\Phi}(t)} |(U(t)\Phi)_i(x)| dx;$$

(b) If $-z_{i\Phi}(t) < h < 0$, then

$$\sigma(h) = \int_0^{z_{i\Phi}(t)} |(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)|dx + \int_{z_{i\Phi}(t)}^{z_{i\Phi}(t)-h} |(U(t)\Phi)_i(x+h)|dx.$$

Since $u_i(x,t)$ is uniformly bounded, the second integral in (a) and the second one in (b) tend to zero uniformly for $\Phi \in B$ when $h \to 0$. Now we introduce a uniform bound for $|(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)|$ depending on h. For $x, x+h < z_{i\Phi}(t)$, we have

$$\begin{aligned}
&|(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)| \\
&\quad = |u_i(x+h,t) - u_i(x,t)| \\
&\quad = \left| \frac{B_i(\tau_i(0;x+h,t))}{g_i(0,P(\tau_i(0;x+h,t)))} \exp\left(-\int_{\tau_i(0;x+h,t)}^t \tilde{m}_i(X_i(s;0,\tau_i(0;x+h,t)),P(s))ds\right)\right. \\
&\qquad \left. -\frac{B_i(\tau_i(0;x,t))}{g_i(0,P(\tau_i(0;x,t)))} \exp\left(-\int_{\tau_i(0;x,t)}^t \tilde{m}_i(X_i(s;0,\tau_i(0;x,t)),P(s))ds\right)\right|.
\end{aligned}$$

To simplify the expressions, let $\hat{\tau}_i = \tau_i(0;x+h,t)$, $\tau_i = \tau_i(0;x,t)$, $\hat{m}_i = \tilde{m}_i(X_i(s;0,\hat{\tau}_i),P(s))$ and $\bar{m}_i = \tilde{m}_i(X_i(s;0,\tau_i),P(s))$. Then we find

$$\begin{aligned}
&|(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)| \\
&\quad = \left| \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} e^{-\int_{\hat{\tau}_i}^t \hat{m}_i ds} - \frac{B_i(\tau_i)}{g_i(0,P(\tau_i))} e^{-\int_{\tau_i}^t \bar{m}_i ds} \right| \\
&\quad \le \left| \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} e^{-\int_{\hat{\tau}_i}^t \hat{m}_i ds} - \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} e^{-\int_{\tau_i}^t \hat{m}_i ds} \right| \\
&\qquad + \left| \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} e^{-\int_{\tau_i}^t \hat{m}_i ds} - \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} e^{-\int_{\tau_i}^t \bar{m}_i ds} \right| \\
&\qquad + \left| \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} e^{-\int_{\tau_i}^t \bar{m}_i ds} - \frac{B_i(\tau_i)}{g_i(0,P(\tau_i))} e^{-\int_{\tau_i}^t \bar{m}_i ds} \right| \\
&\quad \le \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} \left| e^{-\int_{\hat{\tau}_i}^t \hat{m}_i ds} - e^{-\int_{\tau_i}^t \hat{m}_i ds} \right| \\
&\qquad + \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} \left| e^{-\int_{\tau_i}^t \hat{m}_i ds} - e^{-\int_{\tau_i}^t \bar{m}_i ds} \right| \\
&\qquad + \left| \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} - \frac{B_i(\tau_i)}{g_i(0,P(\tau_i))} \right| e^{-\int_{\tau_i}^t \bar{m}_i ds}.
\end{aligned}$$

Let P^+ be the upper bound of P for a fixed t, $\Omega_t := [0, tg_M] \times [0, P^+]$, and let $L_{\tilde{m}_i}$ be the Lipschitz constant for $\tilde{m}_i$ in Ω_t and $\alpha_i := \min\{\min_{(x,P)\in\Omega_t}\{\tilde{m}_i(x,P)\}, 0\}$. Thus, for $h < 0$

($i.e.$ $\tau_i < \hat{\tau}_i$), we obtain

$$
\begin{aligned}
&|(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)| \\
&\quad \le \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} e^{-\alpha_i(t-\tau_i)} \left| \int_{\tau_i}^{\hat{\tau}_i} \hat{m}_i ds \right| \\
&\quad + \frac{B_i(\hat{\tau}_i)}{g_i(0,P(\hat{\tau}_i))} e^{-\alpha_i(t-\tau_i)} L_{\tilde{m}_i} \int_{\tau_i}^{t} |X_i(s;0,\hat{\tau}_i) - X_i(s;0,\tau_i)|\, ds \\
&\quad + A_i^* e^{-\alpha_i(t-\tau_i)} |\hat{\tau}_i - \tau_i|
\end{aligned}
$$

with $A_i^* := \sup_{\tau_i \in [0,t], \Phi_i \in B} \left| \left(\frac{B_i(\tau_i)}{g_i(0,P(\tau_i))} \right)' \right|$.

Let $\tilde{m}_i^* = \max_{(x,P)\in\Omega_t} |\tilde{m}_i(x,P)|$, $(g_i)_x^* = \max_{(x,P)\in\Omega_t} |(g_i)_x(x,P)|$ and $g_i^* = \min_{(x,P)\in\Omega_t} g_i(x,P)$. Then we find the following bounds:

$$
\left| \int_{\tau_i}^{\hat{\tau}_i} \hat{m}_i ds \right| = \left| \int_{\tau_i}^{\hat{\tau}_i} \tilde{m}_i(X_i(s;0,\hat{\tau}_i), P(s)) ds \right| \le \tilde{m}_i^* |\hat{\tau}_i - \tau_i|
$$

and

$$
|X_i(s;0,\hat{\tau}_i) - X_i(s;0,\tau_i)| = |X_i(s;x+h,t) - X_i(s;x,t)| \le |h| e^{t(g_i)_x^*}.
$$

Since τ_i and $\hat{\tau}_i$ satisfy $X_i(t;\tau_i,0) = x$ and $X_i(t;\hat{\tau}_i,0) = x+h$, respectively, we have $X_i(\tau_i;x,t) = 0$ and $X_i(\hat{\tau}_i;x+h,t) = 0$. Then we find

$$
|\hat{\tau}_i - \tau_i| \le |X_i(\hat{\tau}_i;x,t) - X_i(\tau_i;x,t)|/g_i^* = |X_i(\hat{\tau}_i;x,t) - X_i(\hat{\tau}_i;x+h,t)|/g_i^* \le |h| e^{t(g_i)_x^*}/g_i^*.
$$

Hence, we obtain

$$
|(U(t)\Phi)_i(x+h) - (U(t)\Phi)_i(x)| \le e^{((g_i)_x^* - \alpha_i)t} \frac{|h|}{g_i^*} \left\{ B_i(\hat{\tau}_i) \left[\frac{\tilde{m}_i^*}{g_i^*} + L_{\tilde{m}_i} t \right] + A_i^* \right\}. \quad (3.4.4)
$$

Thus, for fixed t and a bounded set of initial conditions, the right hand side of (3.4.4) is uniformly bounded. Also, the lengths of the integration intervals in (a) and (b) are bounded uniformly. Hence $\sigma(h) \to 0$ as $h \to 0^-$. Similarly, we can show that $\sigma(h) \to 0$ as $h \to 0^+$. Thus, $\sigma(h) \to 0$ as $|h| \to 0$. Therefore, $U(t)$ is compact. □

Similarly, the conclusion can be established for $x_{\max} < \infty$.

Lemma 3.4.5. If $W(t)$ is the operator defined by (3.4.2) and $\mu = \inf_{(x,P)\in\Omega, 1\le i\le n} m_i(x,P)$ is strictly positive, then there exists a continuous function $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $k(t,r) \to 0$ as $t \to \infty$ and $\|W(t)\Phi\|_{L^1} \le k(t,r)$ for $\|\Phi\|_{L^1} \le r$.

Proof. By the definition of $W(t)$, we find

$$\begin{aligned}\|W(t)\Phi\|_{L^1} &= \sum_{i=1}^{n}\int_{z_{i\Phi}(t)}^{x_{\max}} u_i(x,t)dx \\ &= \sum_{i=1}^{n}\int_{0}^{x_{\max}} \Phi_i(\xi)e^{-\int_0^t m_i(X_i(s;\xi,0),P(s))ds}d\xi \\ &\leq e^{-\mu t}\|\Phi\|_{L^1}.\end{aligned}$$

Thus, $k(t,r)$ can be taken in the form of $k(t,r) = re^{-\mu t}$. □

Corollary 3.4.1. *Under the hypotheses of lemma 3.4.5, the semigroup $S(t)$ associated with the solutions of problems (3.1.1) is asymptotically smooth.*

Theorem 3.4.6. Let $S(t)$ be the semigroup associated with the solutions of problem (3.1.1) and assume that:
(H6) there exists P^o such that

$$\sup_{(x,P)\in[0,x_{\max})\times[P^o,\infty),1\leq i\leq n} \{\beta_i(x,P) - m_i(x,P)\} = -K < 0.$$

Then $S(t)$ is point dissipative and orbits of bounded sets are bounded.

Proof. In view of (3.2.10), for $P > P^o$, we have $P'(t) \leq nc_M - KP(t)$. Thus, for every bounded set of initial conditions, $P(t) \leq \max\left\{P^o, \frac{nc_M}{K}(1-e^{-Kt}) + \|u_0\|_{L^1}e^{-Kt}\right\}$. Therefore, $S(t)$ is bounded dissipative and so point dissipative, and also orbits of bounded sets are bounded. □

Theorem 3.4.7. We assume that:

$$(H7)\quad \mu = \inf_{(x,P)\in[x_0,x_{\max})\times[0,\infty),1\leq i\leq n} \{m_i(x,P)\} > 0 \ \text{ for some } \ x_0 \in [0,x_{\max}).$$

Then, under the assumptions (H1)-(H7), the C_0-semigroup $S(t)$ defined by the solution of (3.1.1) has a compact global attractor.

Proof. From Corollary 3.4.1 and Theorem 3.4.6, we only need to show that Lemma 3.4.5 (and so corollary 3.4.1) still holds under (H7). In view of the proof of Theorem 3.4.6, $P(t) \leq \max\left\{\frac{nc_M}{K} + \|u_0\|_{L^1}, P^o\right\} := \rho < \infty$. Thus, for $t \geq t(r) :=$

$x_0\left(\min_{(x,P)\in[0,x_0]\times[0,\rho),1\le i\le n} g_i(x,P)\right)^{-1}$ and $\|\Phi\|_{L^1}\le r$, we have $z_{i\Phi}(t)\ge x_0$. Then we find

$$\begin{aligned}\|W(t)\Phi\|_{L^1} &= \sum_{i=1}^{n}\int_{z_{i\Phi}(t)}^{x_{\max}} u_i(x,t)dx\\ &= \sum_{i=1}^{n}\int_{0}^{x_{\max}} \Phi_i(\xi)e^{-\int_0^t m_i(X_i(s;\xi,0),P(s))ds}d\xi\\ &\le \sum_{i=1}^{n}\int_{0}^{x_{\max}} \Phi_i(\xi)e^{-\int_{t_i(r)}^t m_i(X_i(s;\xi,0),P(s))ds}d\xi\\ &\le re^{-\mu(t-t(r))},\end{aligned}$$

provided $\|\Phi\|_{L^1}\le r$. Hence, the proof is completed by letting $k(t,r)=re^{-\mu(t-t(r))}$. □

3.5 Special cases

In this section, we present two special cases under open reproduction (individuals in the ith subpopulation may also reproduce individuals in the jth subpopulation) when $n=2$ and analyze the asymptotic behavior for these cases. To this end, for $i=1,2$, we impose additional assumptions on the parameters as follows:

(H8) m_i and β_i are increasing and nonincreasing functions of P, respectively.

(H9) There is no external inflow of newborns, i.e., $c_i=0$.

3.5.1 Special case A

In the first special case we assume that m_i and β_i depend on the total population P but not on the individual's size. We also assume $x_{\max}=\infty$. Then problem (3.1.1) takes the form:

$$\begin{aligned}&(u_i)_t+(g_i(P(t))u_i)_x+m_i(P(t))u_i=0 && 0<x<\infty,\quad t>0,\\ &g_i(P(t))u_i(0,t)=\sum_{j=1}^{n}\int_0^\infty \gamma_{i,j}\beta_j(P(t))u_j(x,t)dx && t>0,\\ &u_i(x,0)=u_{i0}(x) && 0\le x<\infty.\end{aligned} \tag{3.5.1}$$

Integrating $(3.5.1)_1$ with respect to x and using the boundary condition $(3.5.1)_2$, we obtain the following ordinary differential system:

$$P_i'=\gamma_{i,1}\beta_1(P)P_1+\gamma_{i,2}\beta_2(P)P_2-m_i(P)P_i:=F_i(P_1,P_2) \tag{3.5.2}$$

with $P = P_1 + P_2$.

First, we establish the uniqueness of the positive equilibrium of (3.5.2) when $m_i(P)$ and $\beta_i(P)$ satisfy $m_i(0) < \gamma_{i,i}\beta_i(0)$.

Suppose that $(\bar{P}_1, \bar{P}_2)$ is a positive equilibrium of (3.5.2). Then we have

$$\gamma_{i,1}\beta_1(\bar{P})\bar{P}_1 + \gamma_{i,2}\beta_2(\bar{P})\bar{P}_2 - m_i(\bar{P})\bar{P}_i = 0, \tag{3.5.3}$$

which implies

$$\prod_{i=1}^{2}\left[\gamma_{i,i}\beta_i(\bar{P}) - m_i(\bar{P})\right] = \gamma_{1,2}\gamma_{2,1}\beta_1(\bar{P})\beta_2(\bar{P}) \tag{3.5.4}$$

and

$$m_i(\bar{P}) > \gamma_{i,i}\beta_i(\bar{P}). \tag{3.5.5}$$

Let $h_i(P) = \dfrac{m_i(P)}{\beta_i(P)}$. Then (3.5.4) becomes

$$H(P) = \gamma_{1,2}\gamma_{2,1} \tag{3.5.6}$$

with

$$H(P) = \prod_{i=1}^{2}\left[h_i(\bar{P}) - \gamma_{i,i}\right].$$

On the one hand, since m_i and β_i are increasing and nonincreasing functions of P, respectively, we have that h_i is monotonically increasing. Noting that $m_i(0) < \gamma_{i,i}\beta_i(0)$, we can see that $h_i^{-1}(\gamma_{i,i})$ exists and is unique. Let $\hat{P}_i = h_i^{-1}(\gamma_{i,i})$; then it turns out that $H(\hat{P}_i) = 0$. On the other hand, using (3.5.5) we have $\bar{P} > \hat{P}_i$. Without loss of generality, we may assume $\hat{P}_2 = \max\{\hat{P}_1, \hat{P}_2\}$. Thus we have $\bar{P} > \hat{P}_2$.

Furthermore, we find $H'(P) = \displaystyle\sum_{i,j=1, i\neq j}^{2} h_i'(P)\left[h_j(P) - \gamma_{j,j}\right] > 0$ for $P > \hat{P}_2$. Then $H(\hat{P}_2) = 0$ implies that $H(P) = \gamma_{1,2}\gamma_{2,1}$ has a unique positive solution P satisfying $P > \hat{P}_2$. Thus (3.5.3) has a unique positive solution $\bar{P}$, and it follows from (3.5.3) that (3.5.2) has a unique positive equilibrium $(\bar{P}_1, \bar{P}_2)$.

Moreover, we can show that the unique positive equilibrium $(\bar{P}_1, \bar{P}_2)$ is locally asymptotically stable. For this purpose, let J be the Jacobian matrix evaluated at the equilibrium $(\bar{P}_1, \bar{P}_2)$, that is,

$$J|_{(\bar{P}_1,\bar{P}_2)} = \begin{vmatrix} \gamma_{1,1}\beta_1'\bar{P}_1 + \gamma_{1,2}\beta_2'\bar{P}_2 - m_1'\bar{P}_1 + \gamma_{1,1}\beta_1 - m_1 & \gamma_{1,1}\beta_1'\bar{P}_1 + \gamma_{1,2}\beta_2'\bar{P}_2 - m_1'\bar{P}_1 + \gamma_{1,2}\beta_2 \\ \gamma_{2,1}\beta_1'\bar{P}_1 + \gamma_{2,2}\beta_2'\bar{P}_2 - m_2'\bar{P}_2 + \gamma_{2,1}\beta_1 & \gamma_{2,1}\beta_1'\bar{P}_1 + \gamma_{2,2}\beta_2'\bar{P}_2 - m_2'\bar{P}_2 + \gamma_{2,2}\beta_2 - m_2 \end{vmatrix}.$$

Then we find

$$Tr(J) = \sum_{i,j=1,j\neq i}^{2} \left[\gamma_{i,i}\beta_i'(\bar{P})\bar{P}_i + \gamma_{j,i}\beta_i'(\bar{P})\bar{P}_i - m_i'(\bar{P})\bar{P}_i\right] + \sum_{i=1}^{2} \left[\gamma_{i,i}\beta_i(\bar{P}) - m_i(\bar{P})\right] < 0$$

and

$$\begin{aligned} det(J) = & \sum_{i,j=1,j\neq i}^{2} \left[\gamma_{i,i}\beta_i'(\bar{P})\bar{P}_i + \gamma_{i,j}\beta_j'(\bar{P})\bar{P}_j - m_i'(\bar{P})\bar{P}_i\right]\left[\gamma_{j,j}\beta_j(\bar{P}) - m_j(\bar{P}) - \gamma_{j,i}\beta_i(\bar{P})\right] \\ & + \prod_{i=1}^{2}\left[\gamma_{i,i}\beta_i(\bar{P}) - m_i(\bar{P})\right] - \gamma_{1,2}\gamma_{2,1}\beta_1(\bar{P})\beta_2(\bar{P}). \end{aligned}$$

Noting that $\gamma_{i,i}\beta_i(\bar{P}) < m_i(\bar{P}) < m_i(\bar{P}) + \gamma_{i,j}\beta_j(\bar{P})$, we have

$$det(J) > \prod_{i=1}^{2}\left[\gamma_{i,i}\beta_i(\bar{P}) - m_i(\bar{P})\right] - \gamma_{1,2}\gamma_{2,1}\beta_1(\bar{P})\beta_2(\bar{P}) = 0.$$

Therefore, the unique positive equilibrium $(\bar{P}_1, \bar{P}_2)$ is locally asymptotically stable.

3.5.2 Special case B

In the second special case, we assume that $x_{\max} = 1$ and the ingested food is allocated among maintenance, individual growth and reproduction. In particular, we assume that a fraction $k_i \in (0,1)$ of ingested food is channelled to growth and maintenance, and a fraction $(1-k_i)$ to reproduction. We also assume that for an individual of size x in the ith subpopulation, $k_i f_i(P)x$ is the rate at which maintenance needs energy and $k_i f_i(P)(1-x)$ is what remains for growth. Thus we have the following sub-models for the growth and reproduction rates for each subpopulation

$$g_i(x,P) = g_i^0 k_i f_i(P)(1-x) \quad \text{and} \quad \beta_i(x,P) = \beta_i^0(1-k_i)f_i(P)x,$$

where g_i^0 and β_i^0 are positive constants. In addition, we assume that the mortality rate for an individual in the ith subpopulation is given by $m_i(P)$. Thus problem (3.1.1) becomes

$$\begin{aligned} & (u_i)_t + g_i^0 k_i f_i(P)((1-x)u_i)_x + m_i(P)u_i = 0 && 0 < x \leq 1, \quad t > 0, \\ & g_i^0 k_i u_i(0,t) = \sum_{j=1}^{2} \gamma_{i,j}\beta_j^0(1-k_j)\int_0^1 x u_j(x,t)dx && t > 0, \\ & u_i(x,0) = u_{i0}(x) && 0 \leq x \leq 1. \end{aligned} \tag{3.5.7}$$

Integrating (3.5.7) and multiplying (3.5.7) by x and integrating once again, we have the following system of differential equations:

$$\begin{aligned} P_i' &= \sum_{j=1}^{2} \gamma_{i,j}\beta_j^0(1-k_j)f_j(P)Q_j - m_i(P)P_i, \\ Q_i' &= g_i^0 k_i f_i(P)(P_i - Q_i) - m_i(P)Q_i, \end{aligned} \tag{3.5.8}$$

where $P_i = \int_0^1 u_i(x,t)dx$ and $Q_i = \int_0^1 xu_i(x,t)dx$.

On the one hand, we show that (3.5.8) has a unique positive equilibrium when $m_i(P)$ and $f_i(P)$ satisfy $m_i(0) < \alpha_i f_i(0)$ with $\alpha_i = \dfrac{-g_i^0 k_i + \sqrt{(g_i^0 k_i)^2 + 4g_i^0 k_i \gamma_{i,i}\beta_i^0(1-k_i)}}{2}$.

Suppose that $(\bar{P}_1, \bar{P}_2, \bar{Q}_1, \bar{Q}_2)$ is a positive equilibrium of (3.5.8). Then we have

$$\begin{aligned} &\sum_{j=1}^{2} \gamma_{i,j}\beta_j^0(1-k_j)f_j(\bar{P})\bar{Q}_j - m_i(\bar{P})\bar{P}_i = 0, \\ &g_i^0 k_i f_i(\bar{P})(\bar{P}_i - \bar{Q}_i) - m_i(\bar{P})\bar{Q}_i = 0. \end{aligned} \tag{3.5.9}$$

For simplicity, let $\beta_{i,j} = \gamma_{i,j}\beta_j^0(1-k_j)$ and $g_i = g_i^0 k_i$. Then we obtain

$$\begin{aligned} &\sum_{j=1}^{2} \beta_{i,j} f_j(\bar{P})\bar{Q}_j - m_i(\bar{P})\bar{P}_i = 0, \\ &g_i f_i(\bar{P})(\bar{P}_i - \bar{Q}_i) - m_i(\bar{P})\bar{Q}_i = 0. \end{aligned} \tag{3.5.10}$$

Let $h_i(P) = \dfrac{m_i(P)}{f_i(P)}$. Then from $(3.5.10)_2$ we find

$$\bar{P}_i = \left(\frac{h_i(\bar{P})}{g_i} + 1\right)\bar{Q}_i. \tag{3.5.11}$$

It follows from $(3.5.10)_1$ that

$$\sum_{j=1}^{2} \beta_{i,j} f_j(\bar{P})\bar{Q}_j - m_i(\bar{P})\left(\frac{h_i(\bar{P})}{g_i} + 1\right)\bar{Q}_i = 0, \tag{3.5.12}$$

which implies

$$\prod_{i=1}^{2}\left[\beta_{i,i} f_i(\bar{P}) - m_i(\bar{P})\left(\frac{h_i(\bar{P})}{g_i} + 1\right)\right] = \beta_{1,2}\beta_{2,1} f_1(\bar{P}) f_2(\bar{P}) \tag{3.5.13}$$

and

$$\beta_{i,i} f_i(\bar{P}) < m_i(\bar{P})\left(\frac{h_i(\bar{P})}{g_i} + 1\right). \tag{3.5.14}$$

Let $\varphi_i(P) = \beta_{i,i} f_i(P) - m_i(P)\left(\dfrac{h_i(P)}{g_i} + 1\right)$. Then (3.5.13) and (3.5.14) become

$$\prod_{i=1}^{2} \frac{\varphi_i(\bar{P})}{f_i(\bar{P})} = \beta_{1,2}\beta_{2,1} \tag{3.5.15}$$

and

$$h_i^2(\bar{P}) + g_i h_i(\bar{P}) - g_i\beta_{i,i} > 0, \tag{3.5.16}$$

respectively. Since β_i is a nonincreasing function of P, f_i is a nonincreasing function of P. Note that since m_i is an increasing function of P, we find that h_i is monotonically increasing. Then it follows from $m_i(0) < \alpha_i f_i(0)$ that $h_i^{-1}(\alpha_i)$ exists and is unique. Let $\tilde{P}_i = h_i^{-1}(\alpha_i)$. Without loss of generality, we may assume that $\tilde{P}_2 = \max\{\tilde{P}_1, \tilde{P}_2\}$. Then we find $\bar{P} > \tilde{P}_2$, $\varphi_2(\tilde{P}_2) = 0$ and $\varphi_i(P) < 0$ for $P > \tilde{P}_2$.

Let $I(P) = \prod_{i=1}^{2} \dfrac{\varphi_i(P)}{f_i(P)}$. Then $I'(P) = \dfrac{(\varphi_1'\varphi_2 + \varphi_1\varphi_2')f_1 f_2 - \varphi_1\varphi_2(f_1' f_2 + f_1 f_2')}{f_1^2 f_2^2}$. By the monotonicity of m_i and f_i, we find $\varphi_i'(P) < 0$, which leads to $I'(P) > 0$ for $P > \tilde{P}_2$. Meanwhile, $\varphi_2(\tilde{P}_2) = 0$ yields $I(\tilde{P}_2) = 0$. Hence, $I(P) = \beta_{1,2}\beta_{2,1}$ has a unique solution P satisfying $P > \tilde{P}_2$, which implies that (3.5.12) has a unique solution $\bar{P}$. From (3.5.12), we have $\dfrac{\bar{Q}_1}{\bar{Q}_2}$ is unique. Then by (3.5.11), we have $\dfrac{\bar{P}_1}{\bar{P}_2}$ is unique. Thus $\bar{P}_1$ and $\bar{P}_2$ are unique. Again from (3.5.11) we can see that $\bar{Q}_1$ and $\bar{Q}_2$ are unique. Therefore, (3.5.8) has a unique positive equilibrium $(\bar{P}_1, \bar{P}_2, \bar{Q}_1, \bar{Q}_2)$.

On the other hand, it can be easily shown that if the parameters satisfy $g_i f_i^2 > (m_i' f_i - m_i f_i')I^{-1}(\beta_{12}\beta_{21})$ and $\beta_{i,1}f_1 + \beta_{i,2}f_2 < m_i$, the Jacobian matrix J evaluated at the equilibrium $(\bar{P}_1, \bar{P}_2, \bar{Q}_1, \bar{Q}_2)$ satisfies $a_{kk} < -r_k$ with $r_k = \sum_{l=1, l\neq k}^{n} |a_{kl}|$ for $k = 1, 2, 3, 4$. Then Gerschgorin's circle theorem implies that the eigenvalues of J are negative or have negative real parts. Thus the unique positive equilibrium $(\bar{P}_1, \bar{P}_2, \bar{Q}_1, \bar{Q}_2)$ is locally asymptotically stable.

Remark 1. For the special case B, we found conditions to guarantee that the unique positive equilibrium is locally asymptotically stable. However, our numerous numerical results indicate that the unique positive equilibrium is globally stable. Our future efforts will focus on studying the stability of this unique positive equilibrium.

Bibliography

[1] A.S. Ackleh, H.T. Banks and K. Deng, A finite difference approximation for a coupled system of nonlinear size-structured populations, *Nonlinear Analysis.* **50** (2002), 727-748.

[2] A.S. Ackleh and K. Deng, A monotone approximation for the nonautonomous size-structured population model, *Quart. Appl. Math.* **57** (1999), 261-267.

[3] A.S. Ackleh and K. Deng, A monotone approximation for a nonlinear nonautonomous size-structured population model, *Appl. Math. Comput.* **108** (2000), 103-113.

[4] A.S. Ackleh, K. Ito, An implicit finite difference scheme for the nonlinear size stucture model, *Numer. Funct. Anal. Optim.* **18** (1997), 865-884.

[5] A.S. Ackleh, K. Deng and X. Wang, Competitive exclusion and coexistence for a quasilinear size-structured population model, *Mathematical Biosciences.* **192** (2004), 177-192.

[6] H.T. Banks, L.W. Botsford, F. Kappel and C. Wang, Modeling and estimation in size-structured population models, *Proceedings 2nd Course on Mathematical Ecology (Trieste, 1986)*, World Press, Singapore, 1988, 521-541.

[7] H.T. Banks and B.G. Fitzpatrick, Estimation of growth rate distributions in size-structured population models, *Quart. Appl. Math.* **49** (1991), 215-235.

[8] H.T. Banks, B.G. Fitzpatrick, L.K. Potter and Y. Zhang, Estimation of probability distribution for individual parameters using aggregate population observation, *Stochastic Analysis, Control, Optimization and Applications (W. Mceneaney, G. Yin, Q. Zhang, eds.)*, Birkhäuser, 1998, 353-371.

[9] H.T. Banks and F. Kappel, Transformation semigroups and L^1-approximation for size structured population model, *Semigroup Forum.* **38** (1989), 141-155.

[10] H.T. Banks, F. Kappel and C. Wang, Weak solutions and differentiability for size structured population models, *Birkhäuser Internat. Ser. Numer. Math.* **100** (1991), 35-50.

[11] A. Calsina and J. Saldaña, A model of physiologically structured population dynamics with a nonlinear individual growth rate, *J. Math. Biol.* **33** (1995), 335-364.

[12] J.M. Cushing, A competition model for size-structured species, *SIAM J. Appl. Math.* **49** (1989), 838-858.

[13] N. Dunford and J.T. Schwartz, Linear operators, Part I: General theory, *Interscience Pub., New York.* 1958.

[14] B.G. Fitzpatrick, Modeling and estimation problems for structured heterogeneous populations, *J. Math. Anal.* **172** (1993), 73-91.

[15] J.K. Hale, Asymptotic behavior of dissipative systems, *Mathematical Surveys and Monographs 25. AMS Providence.* 1988.

[16] K. Ito, F. Kappel and G. Peichl, A fully discretized approximation scheme for size-structured population models, *SIAM J. Numer. Anal.* **28** (1991), 923-954.

[17] G.F. Webb, Theory of nonlinear age-dependent population dynamics, Marcel Dekker, New York, 1985.

Appendix: Local existence and uniqueness results

In this part, we establish local existence and uniqueness results for problem (3.1.1).

Lemma 2.2. Suppose that hypotheses (H1)-(H5) hold. Then there exists a value $T > 0$ for which $\mathcal{P}$ has a unique fixed point.

Proof. We use the contraction mapping argument to complete this proof. We first show that $\mathcal{P}$ maps $S_{T,K}$ into itself. In view of (H3) and (H4), it follows from (3.2.6) that

$$B_i(t) \le c_M + \beta_M \sum_{j=1}^{n} \int_0^t B_j(\eta)d\eta + \beta_M \|u_0\|_{L^1}. \tag{$A1$}$$

We then obtain

$$\sum_{j=1}^{n} B_j(t) \le nc_M + n\beta_M \int_0^t \sum_{j=1}^{n} B_j(\eta)d\eta + n\beta_M \|u_0\|_{L^1},$$

which by Gronwall's inequality leads to

$$\sum_{j=1}^{n} B_j(t) \le (nc_M + n\beta_M \|u_0\|_{L^1})\, e^{n\beta_M t}. \tag{$A2$}$$

Making use of (A1) and (A2) yields

$$\begin{aligned}
\mathcal{P}(P)(t) &\le \int_0^t \sum_{j=1}^{n} B_j(\eta)d\eta + ||u_0||_{L^1} \\
&\le (nc_M + n\beta_M \|u_0\|_{L^1}) \int_0^t e^{n\beta_M \eta} d\eta + ||u_0||_{L^1} \\
&\le \frac{c_M}{\beta_M}\left(e^{n\beta_M T} - 1\right) + e^{n\beta_M T} ||u_0||_{L^1} \le K,
\end{aligned}$$

provided T is very small. Thus, $\mathcal{P}$ maps $S_{T,K}$ into itself.

Now, we show that $\mathcal{P}$ is contractive. For any $P, \hat{P} \in S_{T,K}$, let B_i and $\hat{B}_i$ be the solutions of (3.2.6) associated with P and $\hat{P}$, respectively. Below we use the following notations to simplify the expressions:

$$m_{ip} := m_i(X_{ip}(s;\xi,0), P(s)), \quad m_{i\hat{p}} := m_i(X_{i\hat{p}}(s;\xi,0), \hat{P}(s));$$

$$\bar{m}_{ip} := m_i(X_{ip}(s;0,\eta), P(s)), \quad \bar{m}_{i\hat{p}} := m_i(X_{i\hat{p}}(s;0,\eta), \hat{P}(s));$$

$$\beta_{ip} := \beta_i(X_{ip}(t;\xi,0), P(t)), \quad \beta_{i\hat{p}} := \beta_i(X_{i\hat{p}}(t;\xi,0), \hat{P}(t));$$

$$\bar{\beta}_{ip} := \beta_i(X_{ip}(t;0,\eta), P(t)), \quad \bar{\beta}_{i\hat{p}} := \beta_i(X_{i\hat{p}}(t;0,\eta), \hat{P}(t)).$$

Then we find

$$\begin{aligned}
|\mathcal{P}(P)(t) - \mathcal{P}(\hat{P})(t)| &= \Bigg| \sum_{i=1}^{n} \int_0^t B_i(\eta) e^{-\int_\eta^t \bar{m}_{ip} ds}\, d\eta - \sum_{i=1}^{n} \int_0^t \hat{B}_i(\eta) e^{-\int_\eta^t \bar{m}_{i\hat{p}} ds} d\eta \\
&\quad + \sum_{i=1}^{n} \int_0^{x_{\max}} u_{i0}(\xi) \left[e^{-\int_0^t m_{ip} ds} - e^{-\int_0^t m_{i\hat{p}} ds} \right] d\xi \Bigg| \\
&\le \sum_{i=1}^{n} \int_0^t \left| B_i(\eta) - \hat{B}_i(\eta) \right| d\eta \qquad (A3) \\
&\quad + \sum_{i=1}^{n} \int_0^t \hat{B}_i(\eta) \int_\eta^t |\bar{m}_{ip} - \bar{m}_{i\hat{p}}|\, ds d\eta \\
&\quad + \sum_{i=1}^{n} \int_0^{x_{\max}} u_{i0}(\xi) \int_0^t |m_{ip} - m_{i\hat{p}}|\, ds d\xi.
\end{aligned}$$

We now estimate each integral in the last expression of (A3). Let $|F_i(t)| = |B_i(t) - \hat{B}_i(t)|$. Then from (3.2.6) we obtain

$$\begin{aligned}
|F_i(t)| &\le \sum_{j=1}^{n} \left| \int_0^t \gamma_{i,j} \bar{\beta}_{jp} B_j(\eta) e^{-\int_\eta^t \bar{m}_{jp} ds} d\eta - \int_0^t \gamma_{i,j} \bar{\beta}_{j\hat{p}} \hat{B}_j(\eta) e^{-\int_\eta^t \bar{m}_{j\hat{p}} ds} d\eta \right| \\
&\quad + \sum_{j=1}^{n} \left[\int_0^{x_{\max}} \gamma_{i,j} \left| \beta_{jp} e^{-\int_0^t m_{jp} ds} - \beta_{j\hat{p}} e^{-\int_0^t m_{j\hat{p}} ds} \right| u_{j0}(\xi) d\xi \right] \\
&\le \sum_{j=1}^{n} \beta_M \int_0^t \left| B_j(\eta) - \hat{B}_j(\eta) \right| d\eta \\
&\quad + \sum_{j=1}^{n} \beta_M \int_0^t \hat{B}_j(\eta) \int_\eta^t |\bar{m}_{jp} - \bar{m}_{j\hat{p}}|\, ds d\eta \\
&\quad + \sum_{j=1}^{n} \int_0^t \hat{B}_j(\eta) |\bar{\beta}_{jp} - \bar{\beta}_{j\hat{p}}| d\eta \\
&\quad + \sum_{j=1}^{n} \int_0^{x_{\max}} \left| \beta_{jp} e^{-\int_0^t m_{jp} ds} - \beta_{j\hat{p}} e^{-\int_0^t m_{j\hat{p}}} ds \right| u_{j0}(\xi) d\xi.
\end{aligned}$$

We then have

$$|F_i(t)| \le \sum_{j=1}^{n} \beta_M \int_0^t |F_j(\eta)| d\eta + \Psi_i(t), \qquad (A4)$$

where

$$\Psi_i(t) = \sum_{j=1}^{n} \beta_M \int_0^t \hat{B}_j(\eta) \int_\eta^t |\bar{m}_{jp} - \bar{m}_{j\hat{p}}| \, ds d\eta$$
$$+ \sum_{j=1}^{n} \int_0^t \hat{B}_j(\eta) |\bar{\beta}_{jp} - \bar{\beta}_{j\hat{p}}| d\eta$$
$$+ \sum_{j=1}^{n} \int_0^{x_{\max}} \left| \beta_{jp} e^{-\int_0^t m_{jp} ds} - \beta_{j\hat{p}} e^{-\int_0^t m_{j\hat{p}} ds} \right| u_{j0}(\xi) d\xi.$$

Since $X_{jp}(t;0,\eta)$ and $X_{j\hat{p}}(t;0,\eta)$ are the solutions of the equations

$$\begin{cases} \dfrac{d}{dt} x(t) = g_j(x(t), P(t)) \\ x(\eta) = 0 \end{cases} \tag{A5}$$

and

$$\begin{cases} \dfrac{d}{dt} x(t) = g_j(x(t), \hat{P}(t)) \\ x(\eta) = 0 \end{cases} \tag{A6}$$

respectively, we have

$$|X_{jp}(t;0,\eta) - X_{j\hat{p}}(t;0,\eta)| \leq L_{g_j} \int_\eta^t \left(|X_{jp}(s;0,\eta) - X_{j\hat{p}}(s;0,\eta)| + |P(s) - \hat{P}(s)| \right) ds,$$

which yields

$$|X_{jp}(t;0,\eta) - X_{j\hat{p}}(t;0,\eta)| \leq L_{g_j} \int_\eta^t e^{L_{g_j}(t-s)} |P(s) - \hat{P}(s)| ds.$$

Let $L_\beta = \max_{1 \leq i \leq n} L_{\beta_i}$, $L_g = \max_{1 \leq i \leq n} L_{g_i}$ and $L_m = \max_{1 \leq i \leq n} L_{m_i}$. Then we have

$$\begin{aligned} |\bar{m}_{jp} - \bar{m}_{j\hat{p}}| &\leq L_{m_j} \left(|X_{jp}(s;0,\eta) - X_{j\hat{p}}(s;0,\eta)| + |P(s) - \hat{P}(s)| \right) \\ &\leq L_{m_j} \left(1 + L_{g_j} T e^{L_{g_j} T} \right) \|P - \hat{P}\|_\infty \\ &\leq L_m \left(1 + L_g T e^{L_g T} \right) \|P - \hat{P}\|_\infty. \end{aligned}$$

Similarly, we have

$$|\bar{\beta}_{jp} - \bar{\beta}_{j\hat{p}}| \leq L_\beta \left(1 + L_g T e^{L_g T} \right) \|P - \hat{P}\|_\infty.$$

Therefore,

$$\begin{aligned} \Psi_i(t) &\leq \left(1 + L_g T e^{L_g T} \right) \left(\beta_M T L_m + L_\beta \right) \left[T \left(n c_M + n \beta_M \|u_0\|_{L^1} \right) e^{n \beta_M T} + \|u_0\|_{L^1} \right] \|P - \hat{P}\|_\infty \\ &:= J(T) \|P - \hat{P}\|_\infty. \end{aligned}$$

Then it follows from (A4) that

$$|F_i(t)| \le \sum_{j=1}^{n} \beta_M \int_0^t |F_j(\eta)| d\eta + J(T)\|P - \hat{P}\|_\infty.$$

Summing the above inequality over the indices $i = 1, 2, \dots, n$, we find

$$\sum_{i=1}^{n} |F_i(t)| \le n\beta_M \int_0^t \sum_{j=1}^{n} |F_j(\eta)| d\eta + nJ(T)\|P - \hat{P}\|_\infty,$$

which by Gronwall's inequality yields

$$\sum_{i=1}^{n} |F_i(t)| \le nJ(T)e^{n\beta_M T}\|P - \hat{P}\|_\infty.$$

Thus, we have

$$\sum_{i=1}^{n} \int_0^t \left| B_i(\eta) - \hat{B}_i(\eta) \right| d\eta \le nJ(T)e^{n\beta_M T} T\|P - \hat{P}\|_\infty.$$

Arguing analogously, we obtain

$$\begin{aligned} &\sum_{i=1}^{n} \int_0^t \hat{B}_i(\eta) \int_\eta^t |\bar{m}_{ip} - m_{i\hat{p}}| \, ds d\eta \\ &\quad \le L_m T^2 \left(1 + L_g T e^{L_g T}\right) \left(nc_M + n\beta_M \|u_0\|_{L^1}\right) e^{n\beta_M T} \|P - \hat{P}\|_\infty \end{aligned}$$

and

$$\sum_{i=1}^{n} \int_0^{x_{\max}} u_{i0}(\xi) \int_0^t |m_{ip} - m_{i\hat{p}}| \, ds d\xi \le L_m T \left(1 + L_g T e^{L_g T}\right) ||u_0||_{L^1} \|P - \hat{P}\|_\infty.$$

Hence, $\mathcal{P}$ is contractive provided that T is sufficiently small. □

On the other hand, the uniqueness of the solution $P(t)$ and $B_i(t)$ of system (3.2.5)-(3.2.6) implies that the uniqueness of the solution to problem (3.1.1) because each $u_i(x, t)$ given by (3.2.4) is uniquely determined by $P(t)$ and $B_i(t)$. Thus, we have the following local existence result.

Theorem 2.3. Suppose that hypotheses (H1)-(H5) hold. Then there exists a value $T > 0$ such that problem (3.1.1) has a unique solution up to time T.

Conclusion and Future Work

In Chapter 2, we obtained the local existence result for the phytoplankton-zooplankton prey-predator system with phytoplankton aggregation. We wish to develop a new method to study the global existence result. Notice that we used monotone approximation to study this model. We wish to develop an efficient numerical scheme in our future work to corroborate the theoretical results.

An initial-boundary value problem that describes the dynamics of coupled size-structured subpopulations with nonlinear growth, reproduction and mortality rates depending on the individual's size and the total population was discussed in Chapter 3. We generalized the existence and uniqueness of solutions using the contraction mapping argument. We also discussed the asymptotic behavior of solutions to the model using the asymptotic theory of dissipative systems. We show that the unique positive equilibrium is locally asymptotically stable for two open reproduction cases. Our future efforts will focus on further studying the stability of the unique positive equilibrium.

Bibliography

[1] A.S. Ackleh and K. Deng, On a first order hyperbolic coagulation model, *Math. Methods Appl. Sci.* **26** (2003), 703–715.

[2] A.S. Ackleh and B.G. Fitzpatrick, Modeling aggregation and growth processes in an algal population model: analysis and computions, *J. Math. Biol.* **35** (1997), 480-502.

[3] A.S. Ackleh and R.R. Ferdinand, A finite difference approximation for a nonlinear size-structured phytoplankton aggregation model, *Quart. of Appl. Math.* **57** (1999), 501-520.

[4] A.S. Ackleh, K. Ito, An implicit finite difference scheme for the nonlinear size stucture model, *Numer. Funct. Anal. Optim.* **18** (1997), 865-884.

[5] A.S. Ackleh, H.T. Banks and K. Deng, A difference approximation for a coupled system of nonlinear size-structured populations, *Nonlinear Analysis.* **50** (2002), 727-748.

[6] A. Calsina and J. Saldaña, A model of physiologically structured population dynamics with a nonlinear individual growth rate, *J. Math. Biol.* **33** (1995), 335-364.

[7] D.J. McLaughlin, W. Lamb, and A.C. McBride, A semigroup approach to fragmentation models, *SIAM J. Math. Anal.* **28** (1997), 1158–1172.

[8] D.J. McLaughlin, W. Lamb, and A.C. McBride, An existence and uniqueness result for a coagulation and multiple-fragmentation equation, *SIAM J. Math. Anal.* **28** (1997), 1173–1190.

[9] D.J. McLaughlin, W. Lamb, and A.C. McBride, Existence and uniqueness results for the non-autonomous coagulation and multiple-fragmentation equation, *Math. Methods Appl. Sci.* **21** (1998), 1067–1084.

Wang, Xubo. Bachelor of Science, Henan University, China, Spring 1998;
Master of Science, Southeast University, China, Spring 2001;
Doctor of Philosophy, University of Louisiana at Lafayette, Summer 2005
Major: Mathematics
Title of Dissertation: Nonlinear Size-Structured Population Models
Dissertation Directors: Dr. Azmy S. Ackleh, Dr. Keng Deng
Pages in Dissertation: 80; Words in Abstract: 216

ABSTRACT

In this dissertation, some size-structured population models are studied.

In Chapter 1, a quasilinear size-structured model that describes the dynamics of a population with n competing ecotypes is studied. Under the assumption that the vital rates of each subpopulation depend on the total population due to competition, the conditions on the individual rates which guarantee competitive exclusion in the case of closed reproduction are provided. In particular, the results suggest that the ratio of the reproduction and mortality rates is a good measure to determine the winning ecotype. Meanwhile, in the case of open reproduction, the coexistence of all ecotypes is established.

In Chapter 2, a model that describes the dynamics of the phytoplankton and zooplankton prey-predator system within the context of phytoplankton aggregation is considered. Existence-uniqueness results of the solution are established via a comparison principle and the upper-lower solution technique.

In Chapter 3, a size-structured model that describes the dynamics of n-subpopulations with nonlinear growth, reproduction and mortality rates is investigated. Existence and uniqueness results for the solutions are established. The existence of a compact global attractor for the trajectories of the dynamical system defined by the solutions of this model is also obtained. In addition, two special cases under open reproduction are studied and the asymptotic dynamics for these special cases are analyzed.

BIOGRAPHICAL SKETCH

Xubo Wang was born in September 1977, in Nanyang, Henan Province, China. She was admitted to Henan University in August 1994 and received her Bachelor of Science in Mathematics Education in July 1998. Then she earned admittance to the graduate school at Southeast University in August 1998 and received her degree of Master of Science in Applied Mathematics in July 2001. In August 2001, she joined the Ph.D. program in the Department of Mathematics at the University of Louisiana at Lafayette. She completed the requirements for the degree Doctor of Philosophy in July 2005.