Structured Population Models: Well-Posedness, Approximation and Parameter Estimation

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## Preface

In biological population, individuals may differ in a 'structure' variable such as age, size, weight and other quantities that influence individual development-growth, reproduction and mortality. Modelling such kind of phenomenon is structured population model, which bridges the gap between mechanisms at the individual level and behavior at the level of the population.

In a typical direct problem one prescribes model ingredients that describe mechanisms at the individual level, lifts the model to the population level, and finally studies phenomena at the population level. In the inverse problem the situation is reversed. Using knowledge about behavior at the population level one wants to deduce the underlying mechanisms at the individual level.

Lots of literature have been contributed to the development of theoretical and computational methods for the direct and inverse problem of different structured population models. However, there are still lots of open problems that need to be explored, which inspire us to study some of them.

A hierarchical size-structured model with nonlinear growth, mortality and reproduction rates is studied in Chapter 1. A finite difference approximation is developed to establish the existence-uniqueness of the weak solution to the model. Simulations indicate that the monotonicity assumption on the growth rate is crucial for the global existence of weak solutions to the hierarchical model.

The inverse problem in a coupled system of nonlinear size-structured populations
is studied in Chapter 2. A least-squares technique is developed for identifying unknown parameters and its Convergence results are also established. Ample numerical simulations and statistical evidence are provided to demonstrate the feasibility of this approach.

A nonlinear size-structured phytoplankton-zooplankton aggregation Model is studied in Chapter 3. A monotone approximation method is constructed to establish the existence-uniqueness of the weak solution to the model.

We also provide a numerical solver in this dissertation for the general size-structured population model which is a generalization of the models discussed in Chapter 1 and Chapter 2. This solver was written by using Matlab to be a user-friendly package which was compiled to a stand-alone one. In Chapter 4, we give some instructions about how to use this package and how we compile it into a stand-alone package.

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## Chapter 1

## A Quasilinear Hierarchical Size Structured Model: Well-Posedness and Approximation

In this chapter, a finite difference approximation to a hierarchical size-structured model with nonlinear growth, mortality and reproduction rates is developed. Existence-uniqueness of the weak solution to the model is established and convergence of the finite difference approximation is proved. Simulations indicate that the monotonicity assumption on the growth rate is crucial for the global existence of weak solutions. Numerical results testing the efficiency of this method in approximating the long-time behavior of the model are presented.

### 1.1 Introduction

In this chapter, we consider the following initial-boundary value problem which models the evolution of a hierarchical size-structured population:

$$
\begin{align*}
& u_{t}+(g(x, Q(x, t)) u)_{x}+m(x, Q(x, t)) u=0, \quad(x, t) \in(0, L] \times(0, T] \\
& g(0, Q(0, t)) u(0, t)=C(t)+\int_{0}^{L} \beta(x, Q(x, t)) u(x, t) d x, \quad t \in(0, T]  \tag{1.1.1}\\
& u(x, 0)=u^{0}(x), \quad x \in[0, L] .
\end{align*}
$$

Here $u(x, t)$ is the density of individuals having size $x$ at time $t$, and

$$
Q(x, t)=\alpha \int_{0}^{x} w(\xi) u(\xi, t) d \xi+\int_{x}^{L} w(\xi) u(\xi, t) d \xi, \quad 0 \leq \alpha<1,
$$

is a function of the density $u$ referred to hereafter as the environment. The coefficient $\alpha$ is related to the degree of hierarchy in the population. More precisely, $\alpha$ is the weight of the lower ranks in the competition for resources. The function $w$ represents the population measure being used. For example, $w(x)=1$ means the total number of individuals in the population while $w(x)=x$ means the total biomass. The function $g$ denotes the growth rate of an individual and $m$ denotes the mortality rate of an individual. The function $\beta$ is the reproduction rate of an individual, while $C$ represents the inflow rate of zero-size individual from an external source. It is worth noting that $g, m$ and $\beta$ are functions of both the size and the environment.

Hierarchically structured population models have been studied by several researchers during the past decade (e.g., $[3,4,6,7,8,9]$ ). Below, we briefly discuss the models that are most related to (1.1.1). In [4] the model (1.1.1) was considered under the assumption that the vital rates depend only on the environment $Q(x, t)$, i.e., $g=g(Q)$, $\beta=\beta(Q), m=m(Q)$ and $C=0$. Therein, the authors transform the nonlocal PDE into a local one by means of variable change similar to that used in the age-dependent case [6]. Moreover, a decoupled ordinary differential equation is obtained for the total population. The existence and uniqueness of solutions for the transformed problem are proved, and hence an existence-uniqueness result for the original problem is established (under a compatibility condition on the initial data $u^{0}$ ). However, their method does not apply to the more general setting presented in (1.1.1).

In [3] the model (1.1.1) was studied with parameters $g$, $\beta$ dependent linearly on the size $x, m$ independent of $x$, and $C(t)=0$ as well. The existence-uniqueness of solutions to the model is proved using an equivalent pair of partial and ordinary differential equations,
where the ODE describes the dynamics of the total biomass.
In [9] problem (1.1.1) was investigated with $\alpha=0$. Such a problem is used to model competition for light in a forest, whose distinct feature is its hierarchical nature (see, e.g., [11]). This means that taller trees are overshadowing the smaller ones, but not vice versa. The analysis therein uses a coordinate transform which brings the model into a simple form reducing the first order partial differential equation to a family of coupled ordinary differential equations for population density and size as functions of characteristic variables. An existence-uniqueness result for the coupled ODE is obtained, from which the existence-uniqueness of continuous solutions for the original problem follows (under a compatibility condition on the initial data).

In this chapter, we are also concerned with the existence-uniqueness of solutions to (1.1.1). A framework similar to the one in $[1,2,5,10]$ is used to obtain existenceuniqueness of weak solutions as well as convergence of the difference approximation. There are two main differences between the model (1.1.1) and classical scalar conservation laws such as those considered in $[5,10]$. 1) A scalar conservation law is considered on $\mathbb{R}$ while (1.1.1) is considered on a compact interval $[0, L]$ with a nonlinear nonlocal boundary condition. 2) The flux in a conservation law is a local nonlinearity versus a nonlocal nonlinearity in model (1.1.1). Problem (1.1.1) is also different from the case where $\alpha=1$ considered in $[1,2]$. For this special case of $\alpha$ the environment $Q=$ $Q(t)=\int_{0}^{L} w(\xi) u(\xi, t) d \xi$ is only a function of time. These differences result in different dynamics. In particular, it is well known that without any monotonicity assumption on the flux term in a conservation law or in the model (1.1.1) with $\alpha=1$, a unique bounded solution exists under some regularity assumptions on the parameters. However, this is not the case for the hierarchical structured model (1.1.1) with $0 \leq \alpha<1$. As is shown in Section 1.3, solution to this model may blow up in finite time if $g$ is not monotone. Therefore, in Section 1.2, we develop new techniques to handle these differences and obtain the necessary apriori estimates.

By a weak solution to problem (1.1.1) we mean a bounded and measurable function $u(x, t)$ satisfying

$$
\begin{align*}
\int_{0}^{L} u(x, & t) \varphi(x, t) d x-\int_{0}^{L} u^{0}(x) \varphi(x, 0) d x \\
= & \int_{0}^{t} \int_{0}^{L}\left(u \varphi_{s}+g u \varphi_{x}-m u \varphi\right) d x d s  \tag{1.1.2}\\
& +\int_{0}^{t} \varphi(0, s)\left(C(s)+\int_{0}^{L} \beta(x, Q(x, s)) u(x, s) d x\right) d s
\end{align*}
$$

for $t \in[0, T]$, and every test function $\varphi \in \mathcal{C}^{1}((0, L) \times(0, T))$. We note that this is the first result on convergence of approximation for a hierarchical size-structured model with nonlinear growth, reproduction and mortality rates. From our point of view, this is what gives our approach to establishing existence-uniqueness an advantage over the abovementioned ones, since it results in a numerical scheme which can be used for studying the long-time behavior of the model (see Section 3 for an example). Furthermore, as is seen below, our approach does not require any (biologically irrelevant) compatibility condition on the initial data $u^{0}$.

The following regularity conditions will be imposed on our model parameters throughout this chapter:
(H1) $g(x, Q)$ is twice continuously differentiable with respect to $x$ and $Q, g(x, Q)>0$ for $x \in[0, L)$ and $g(L, Q)=0, g_{Q}(x, Q) \leq 0$.
(H2) $m(x, Q)$ is nonnegative continuously differentiable with respect to $x$ and $Q$.
(H3) $\beta(x, Q)$ is nonnegative continuously differentiable with respect to $x$ and $Q$. Furthermore, there is a constant $\omega_{1}>0$ such that $\sup _{(x, Q) \in[0, L] \times[0, \infty)} \beta(x, Q) \leq \omega_{1}$.
(H4) $w(x)$ is nonnegative continuously differentiable.
(H5) $C(t)$ is nonnegative continuously differentiable.
(H6) $u^{0} \in B V[0, L]$ and $u^{0}(x) \geq 0$.

Smoothness assumptions of similar type with respect to the environment $Q$ have been used in other hierarchical size-structured models (see [4]). Such smoothness in $Q$ seems not to cause any significant restrictions in the intended applications. For the convenience of exposition, we require the same smoothness with respect to $x$, although such smoothness in $x$ can be relaxed. For example, some applications assume that the birth rate $\beta$ is a piecewise continuous function in $x$. This case can be treated by our methodology, however, additional technicalities will be required.

The remainder of this chapter is organized as follows. In Section 1.2, we develop a numerical scheme for computing the solution of (1.1.1) and prove the convergence of this scheme to the bounded total variation unique solution. In Section 1.3, we present a numerical example which shows that the assumption $g_{Q} \leq 0$ is necessary for the global existence of weak solutions. Furthermore, we give another example which demonstrates how well our scheme performs in approximating the long-time behavior of solutions to the model (1.1.1).

### 1.2 Existence-Uniqueness and Convergence of approximation

In this section we establish the existence and uniqueness of weak solutions to (1.1.1). This will be done through the following series of steps: 1) We construct a finite difference approximation for the model (1.1.1). 2) We establish apriori bounds for the solutions to the difference approximation (Lemmas 1.2.1, 1.2.2, 1.2.6 and 1.2.7). 3) These apriori bounds are then used to show that a set of functions generated from the difference approximation is compact in $\mathcal{L}^{1}((0, L) \times(0, T))$ topology, and hence we are able to pass to the limit along a subsequence. This shows the existence of a weak solution. 4) Finally, we prove uniqueness of the weak solution, and hence establish convergence of the difference approximation.

The following notation will be used throughout the paper: $\triangle x=L / n$ and $\triangle t=T / l$ denote the spatial and time mesh sizes, respectively. The mesh points are given by: $x_{j}=j \triangle x, j=0,1, \ldots, n$ and $t_{k}=k \Delta t, k=0,1, \ldots, l$. We denote by $u_{j}^{k}$ and $Q_{j}^{k}$ the finite difference approximations of $u\left(x_{j}, t_{k}\right)$ and $Q\left(x_{j}, t_{k}\right)$, respectively, and we let

$$
g_{j}^{k}=g\left(x_{j}, Q_{j}^{k}\right), \beta_{j}^{k}=\beta\left(x_{j}, Q_{j}^{k}\right), m_{j}^{k}=m\left(x_{j}, Q_{j}^{k}\right), w_{j}=w\left(x_{j}\right), C^{k}=C\left(t_{k}\right)
$$

We define the difference operator

$$
D_{\Delta x}^{-}\left(u_{j}^{k}\right)=\frac{u_{j}^{k}-u_{j-1}^{k}}{\triangle x}, \quad 1 \leq j \leq n
$$

and let the $\ell^{1}$ and $\ell^{\infty}$ norms of $u^{k}$ by

$$
\left\|u^{k}\right\|_{1}=\sum_{j=1}^{n}\left|u_{j}^{k}\right| \triangle x, \quad\left\|u^{k}\right\|_{\infty}=\max _{j}\left|u_{j}^{k}\right|
$$

respectively. Since $g$ is a positive function, we discretize the partial differential equation in (1.1.1) using the following upwind implicit finite difference approximation:

$$
\begin{align*}
& \frac{u_{j}^{k+1}-u_{j}^{k}}{\triangle t}+\frac{g_{j}^{k} u_{j}^{k+1}-g_{j-1}^{k} u_{j-1}^{k+1}}{\triangle x}+m_{j}^{k} u_{j}^{k+1}=0,1 \leq j \leq n \\
& g_{0}^{k} u_{0}^{k+1}=C^{k}+\sum_{j=1}^{n} \beta_{j}^{k} u_{j}^{k} \triangle x  \tag{1.2.1}\\
& Q_{j}^{k}=\alpha \sum_{i=1}^{j} w_{i} u_{i}^{k} \Delta x+\sum_{i=j+1}^{n} w_{i} u_{i}^{k} \Delta x
\end{align*}
$$

with the initial condition

$$
u_{j}^{0}=\frac{1}{\triangle x} \int_{(j-1) \Delta x}^{j \Delta x} u^{0}(x) d x \quad j=1,2, \ldots, n
$$

If we define

$$
d_{j}^{k}=1+\frac{\triangle t}{\triangle x} g_{j}^{k}+\triangle t m_{j}^{k} \quad j=1,2, \ldots, n
$$

then (1.2.1) can be equivalently written as the following system of linear equations for $\vec{u}^{k+1}=\left[u_{0}^{k+1}, u_{1}^{k+1}, \ldots, u_{n}^{k+1}\right]^{T} \in \mathbb{R}^{n+1}$

$$
\begin{equation*}
A^{k} \vec{u}^{k+1}=\vec{f}^{k} \tag{1.2.2}
\end{equation*}
$$

where $\overrightarrow{f^{k}}=\left[C^{k}+\sum_{j=1}^{n} \beta_{j}^{k} u_{j}^{k} \triangle x, u_{1}^{k}, \ldots, u_{n}^{k}\right]^{T}$ and $A^{k}$ is the following lower triangular matrix

$$
A^{k}=\left(\begin{array}{cccccc}
g_{0}^{k} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{\Delta t}{\Delta x} g_{0}^{k} & d_{1}^{k} & 0 & \cdots & 0 & 0 \\
0 & -\frac{\Delta t}{\Delta x} g_{1}^{k} & d_{2}^{k} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & -\frac{\Delta t}{\Delta x} g_{n-1}^{k} & d_{n}^{k}
\end{array}\right) .
$$

We develop the above implicit scheme because for any choice of $\Delta t$ and $\triangle x$, one can easily see that under the assumptions on our parameters, the system (1.2.2) has a unique solution satisfying $\vec{u}^{k+1} \geq 0, k=0,1, \ldots, l-1$. From a biological point of view, it is very important that the numerical approximation preserves the nonnegativity of the solution. Next we state that the difference approximation is bounded in $\ell^{1}$ norm. The proof of this result is similar to that of Lemma 1 in [1], and hence is omitted.

Lemma 1.2.1. The following estimate holds:

$$
\left\|u^{k}\right\|_{1} \leq\left(1+\omega_{1} \triangle t\right)^{k}\left\|u^{0}\right\|_{1}+\sum_{i=1}^{k}\left(1+\omega_{1} \triangle t\right)^{k-i} C^{i-1} \triangle t
$$

and thus

$$
Q_{j}^{k} \leq Q_{\max }=\|w\|_{\infty} \sup _{n, l}\left[\left(1+\omega_{1} \triangle t\right)^{l}\left\|u^{0}\right\|_{1}+\sum_{i=1}^{l}\left(1+\omega_{1} \triangle t\right)^{l-i} C^{i-1} \triangle t\right]<\infty
$$

Remark 1.2.1. Since $g$ is twice continuously differentiable with respect to $x$ and $Q$, and $m$ is continuously differentiable with respect to $x$ and $Q$, there exists a positive constant $\omega_{2}$ such that

$$
\max _{1 \leq j \leq n}\left|\frac{g\left(x_{j}, Q_{j}^{k}\right)-g\left(x_{j-1}, Q_{j}^{k}\right)}{\triangle x}+m\left(x_{j}, Q_{j}^{k}\right)\right| \leq \max _{(x, Q) \in \mathbb{D}}\left|g_{x}(x, Q)\right|+\max _{(x, Q) \in \mathbb{D}}|m(x, Q)|:=\omega_{2},
$$

where $\mathbb{D}=\left\{(x, Q) \mid(x, Q) \in[0, L] \times\left[0, Q_{\max }\right]\right\}$.
We then establish an $\ell^{\infty}$ bound on the difference approximation. Here $g_{Q} \leq 0$ plays a crucial role for establishing such a bound. Note that by (H1) there exists a positive constant $\mu$ such that $\mu \leq g(0, Q)$ for $Q \in\left[0, Q_{\max }\right]$.

Lemma 1.2.2. Assume that $\Delta t$ is chosen to satisfy $\omega_{2} \Delta t<1$. Then we have the estimate

$$
\left\|u^{k}\right\|_{\infty} \leq \max \left\{\left(1-\omega_{2} \triangle t\right)^{-k}\left\|u^{0}\right\|_{\infty}, \frac{\omega_{1}\left\|u^{k-1}\right\|_{1}+C^{k-1}}{\mu}\right\}
$$

Proof. If $u_{0}^{k+1}=\max _{j} u_{j}^{k+1}$, then from the second equation of (1.2.1) we get

$$
g_{0}^{k} u_{0}^{k+1}=C^{k}+\sum_{i=1}^{n} \beta_{i}^{k} u_{i}^{k} \triangle x \leq \omega_{1}\left\|u^{k}\right\|_{1}+C^{k}
$$

Since $\mu \leq g(0, Q)$ for $Q \in\left[0, Q_{\max }\right]$, we obtain

$$
\begin{equation*}
\max _{j} u_{j}^{k+1}=u_{0}^{k+1} \leq \frac{\omega_{1}\left\|u^{k}\right\|_{1}+C^{k}}{g_{0}^{k}} \leq \frac{\omega_{1}\left\|u^{k}\right\|_{1}+C^{k}}{\mu} . \tag{1.2.3}
\end{equation*}
$$

Now, suppose that for some $1 \leq i \leq n, u_{i}^{k+1}=\max _{j} u_{j}^{k+1}$. Then from the first equation of (1.2.1), we have

$$
\left(1+\frac{\Delta t}{\triangle x} g_{i}^{k}+\triangle t m_{i}^{k}\right) u_{i}^{k+1}-\frac{\triangle t}{\triangle x} g_{i-1}^{k} u_{i-1}^{k+1}=u_{i}^{k} .
$$

Since $u_{i-1}^{k+1} \leq u_{i}^{k+1}$, we obtain

$$
\left(1+\triangle t \frac{g_{i}^{k}-g_{i-1}^{k}}{\triangle x}+\triangle t m_{i}^{k}\right) u_{i}^{k+1} \leq u_{i}^{k}
$$

Furthermore, by adding and subtracting alike terms we find that

$$
\begin{equation*}
\frac{g_{i}^{k}-g_{i-1}^{k}}{\triangle x}=\frac{g\left(x_{i}, Q_{i}^{k}\right)-g\left(x_{i-1}, Q_{i}^{k}\right)}{\triangle x}+g_{Q}\left(x_{i-1}, \widetilde{Q}\right) \frac{Q_{i}^{k}-Q_{i-1}^{k}}{\triangle x} \tag{1.2.4}
\end{equation*}
$$

where $\widetilde{Q}$ is between $Q_{i-1}^{k}$ and $Q_{i}^{k}$. Clearly,

$$
\begin{equation*}
\frac{Q_{i}^{k}-Q_{i-1}^{k}}{\triangle x}=\alpha \sum_{j=1}^{i} w_{j} u_{j}^{k}+\sum_{j=i+1}^{n} w_{j} u_{j}^{k}-\alpha \sum_{j=1}^{i-1} w_{j} u_{j}^{k}-\sum_{j=i}^{n} w_{j} u_{j}^{k}=(\alpha-1) w_{i} u_{i}^{k} \tag{1.2.5}
\end{equation*}
$$

Hence,

$$
\left[1+\triangle t\left(m_{i}^{k}+\frac{g\left(x_{i}, Q_{i}^{k}\right)-g\left(x_{i-1}, Q_{i}^{k}\right)}{\triangle x}\right)+\triangle t g_{Q}\left(x_{i-1}, \widetilde{Q}\right)(\alpha-1) w_{i} u_{i}^{k}\right] u_{i}^{k+1} \leq u_{i}^{k} .
$$

Noticing that $g_{Q} \leq 0$ and $0 \leq \alpha<1$, and using Remark 1.2.1, we obtain

$$
\left(1-\omega_{2} \triangle t\right) u_{i}^{k+1} \leq u_{i}^{k}
$$

Since $1-\omega_{2} \triangle t>0$, we have

$$
u_{i}^{k+1} \leq \frac{1}{1-\omega_{2} \Delta t} u_{i}^{k} \leq \frac{1}{1-\omega_{2} \Delta t} \max _{j} u_{j}^{k}
$$

which implies

$$
\begin{equation*}
\max _{j} u_{j}^{k} \leq\left(1-\omega_{2} \Delta t\right)^{-k} \max _{j} u_{j}^{0} \tag{1.2.6}
\end{equation*}
$$

A combination of (1.2.3) and (1.2.6) then yields

$$
\left\|u^{k}\right\|_{\infty} \leq \max \left\{\left(1-\omega_{2} \triangle t\right)^{-k}\left\|u^{0}\right\|_{\infty}, \frac{\omega_{1}\left\|u^{k-1}\right\|_{1}+C^{k-1}}{\mu}\right\}
$$

The next three lemmas are necessary to show that the approximation $u_{j}^{k}$ has bounded total variation.

Lemma 1.2.3. Assume that $\Delta t$ is chosen to satisfy $\omega_{2} \Delta t<1$. Then there exists $a$ positive constant $E_{1}$ such that for each $k$

$$
\begin{equation*}
\max _{i}\left|\frac{Q_{i}^{k}-Q_{i}^{k-1}}{\triangle t}\right| \leq E_{1} \tag{1.2.7}
\end{equation*}
$$

Proof. From the third equation of (1.2.1), we find

$$
\begin{aligned}
\frac{Q_{i}^{k}-Q_{i}^{k-1}}{\Delta t}= & \alpha \sum_{j=1}^{i} w_{j} \frac{u_{j}^{k}-u_{j}^{k-1}}{\Delta t} \Delta x+\sum_{j=i+1}^{n} w_{j} \frac{u_{j}^{k}-u_{j}^{k-1}}{\triangle t} \Delta x \\
= & -\left[\alpha \sum_{j=1}^{i} w_{j} m_{j}^{k-1} u_{j}^{k}+\sum_{j=i+1}^{n} w_{j} m_{j}^{k-1} u_{j}^{k}\right] \triangle x \\
& -\left[\alpha \sum_{j=1}^{i} w_{j} D_{\Delta x}^{-}\left(g_{j}^{k-1} u_{j}^{k}\right)+\sum_{j=i+1}^{n} w_{j} D_{\Delta x}^{-}\left(g_{j}^{k-1} u_{j}^{k}\right)\right] \Delta x .
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& -\left[\alpha \sum_{j=1}^{i} w_{j} D_{\Delta x}^{-}\left(g_{j}^{k-1} u_{j}^{k}\right)+\sum_{j=i+1}^{n} w_{j} D_{\Delta x}^{-}\left(g_{j}^{k-1} u_{j}^{k}\right)\right] \Delta x \\
= & \alpha \sum_{j=1}^{i-1} g_{j}^{k-1} w_{x}\left(\bar{x}_{j+1}\right) u_{j}^{k} \triangle x+\sum_{j=i+1}^{n-1} g_{j}^{k-1} w_{x}\left(\bar{x}_{j+1}\right) u_{j}^{k} \triangle x \\
& +\alpha w_{1} g_{0}^{k-1} u_{0}^{k}-\alpha w_{i} g_{i}^{k-1} u_{i}^{k}+w_{i+1} g_{i}^{k-1} u_{i}^{k}
\end{aligned}
$$

where $\bar{x}_{j+1}$ is between $x_{j}$ and $x_{j+1}$. Hence,

$$
\begin{aligned}
\max _{i}\left|\frac{Q_{i}^{k}-Q_{i}^{k-1}}{\triangle t}\right| \leq & {\left[\max _{(x, Q) \in \mathbb{D}}|m(x, Q)|\|w\|_{\infty}+\max _{(x, Q) \in \mathbb{D}}|g(x, Q)| \max _{x \in[0, L]}\left|w^{\prime}(x)\right|\right]\left\|u^{k}\right\|_{1} } \\
& +3 \max _{(x, Q) \in \mathbb{D}}|g(x, Q)|\|w\|_{\infty}\left\|u^{k}\right\|_{\infty}
\end{aligned}
$$

Thus by Lemmas 1.2 .1 and 1.2 .2 , there exists a positive constant $E_{1}$ such that for each $k \max _{i}\left|\frac{Q_{i}^{k}-Q_{i}^{k-1}}{\triangle t}\right| \leq E_{1}$.

Lemma 1.2.4. Assume that $\Delta t$ is chosen to satisfy $\omega_{2} \Delta t<1$. Then there exist a positive constant $E_{2}$ such that for each $k$ we have

$$
\begin{equation*}
\left|\frac{u_{0}^{k+1}-u_{0}^{k}}{\Delta t}\right| \leq E_{2} \tag{1.2.8}
\end{equation*}
$$

Proof. By the fact that

$$
\frac{g_{0}^{k} u_{0}^{k+1}-g_{0}^{k-1} u_{0}^{k}}{\triangle t}=g_{0}^{k} \frac{u_{0}^{k+1}-u_{0}^{k}}{\triangle t}+u_{0}^{k} \frac{g_{0}^{k}-g_{0}^{k-1}}{\triangle t}
$$

we have from the boundary condition

$$
\begin{align*}
& g_{0}^{k} \frac{u_{0}^{k+1}-u_{0}^{k}}{\triangle t}+u_{0}^{k} g_{0}^{k}-g_{0}^{k-1} \\
\triangle t & \frac{C^{k}-C^{k-1}}{\triangle t}  \tag{1.2.9}\\
= & \sum_{i=1}^{n} \frac{\beta_{i}^{k} u_{i}^{k}-\beta_{i}^{k-1} u_{i}^{k-1}}{\triangle t} \triangle x=\sum_{i=1}^{n}\left[\beta_{i}^{k} \frac{u_{i}^{k}-u_{i}^{k-1}}{\triangle t}+u_{i}^{k-1} \frac{\beta_{i}^{k}-\beta_{i}^{k-1}}{\triangle t}\right] \triangle x \\
= & \sum_{i=1}^{n}\left\{-\beta_{i}^{k}\left[D_{\Delta x}^{-}\left(g_{i}^{k-1} u_{i}^{k}\right)+m_{i}^{k-1} u_{i}^{k}\right]+u_{i}^{k-1} \beta_{Q}\left(x_{i}, \bar{Q}_{i}^{k}\right) \frac{Q_{i}^{k}-Q_{i}^{k-1}}{\triangle t}\right\} \triangle x,
\end{align*}
$$

where $\bar{Q}_{i}^{k}$ is between $Q_{i}^{k}$ and $Q_{i}^{k-1}$. Note that

$$
\begin{aligned}
& \sum_{i=1}^{n}\left[-\beta_{i}^{k} D_{\Delta x}^{-}\left(g_{i}^{k-1} u_{i}^{k}\right)\right] \triangle x=\beta_{1}^{k} g_{0}^{k-1} u_{0}^{k}+\sum_{j=1}^{n-1} g_{j}^{k-1} u_{j}^{k}\left(\beta_{j+1}^{k}-\beta_{j}^{k}\right) \\
& \quad=\beta_{1}^{k} g_{0}^{k-1} u_{0}^{k}+\sum_{j=1}^{n-1} g_{j}^{k-1}\left[\beta_{x}\left(\bar{x}_{j+1}, Q_{j+1}^{k}\right)+(\alpha-1) \beta_{Q}\left(x_{j}, \bar{Q}_{j+1}^{k}\right) w_{j+1} u_{j+1}^{k}\right] u_{j}^{k} \triangle x
\end{aligned}
$$

where $\bar{x}_{j+1} \in\left(x_{j}, x_{j+1}\right)$, and $\bar{Q}_{j+1}^{k}$ is between $Q_{j}^{k}$ and $Q_{j+1}^{k}$. Substituting the above equation in (1.2.9) we have

$$
\begin{aligned}
& \left|g_{0}^{k} \frac{u_{0}^{k+1}-u_{0}^{k}}{\triangle t}+u_{0}^{k} \frac{g_{0}^{k}-g_{0}^{k-1}}{\triangle t}-\frac{C^{k}-C^{k-1}}{\triangle t}\right| \\
\leq & \omega_{1} \max _{(x, Q) \in \mathbb{D}}|m(x, Q)|\left\|u^{k}\right\|_{1}+E_{1} \max _{(x, Q) \in \mathbb{D}}\left|\beta_{Q}(x, Q)\right|\left\|u^{k-1}\right\|_{1}+\omega_{1} \max _{(x, Q) \in \mathbb{D}} g(x, Q)\left\|u^{k}\right\|_{\infty} \\
& +\max _{(x, Q) \in \mathbb{D}} g(x, Q)\left[\max _{(x, Q) \in \mathbb{D}}\left|\beta_{x}(x, Q)\right|+(1-\alpha) \max _{(x, Q) \in \mathbb{D}}\left|\beta_{Q}(x, Q)\right|\|w\|_{\infty}\left\|u^{k}\right\|_{\infty}\right]\left\|u^{k}\right\|_{1} .
\end{aligned}
$$

Note that

$$
\frac{g_{0}^{k}-g_{0}^{k-1}}{\triangle t}=\frac{g\left(0, Q_{0}^{k}\right)-g\left(0, Q_{0}^{k-1}\right)}{\triangle t}=g_{Q}\left(0, \bar{Q}_{0}^{k}\right) \frac{Q_{0}^{k}-Q_{0}^{k-1}}{\triangle t}
$$

where $\bar{Q}_{0}^{k}$ is between $Q_{0}^{k-1}$ and $Q_{0}^{k}$. Hence, by (1.2.7) we have $\left|\frac{g_{0}^{k}-g_{0}^{k-1}}{\triangle t}\right| \leq E_{1} \max _{(x, Q) \in \mathbb{D}}\left|g_{Q}(x, Q)\right|$. Thus by (H5), $\mu \leq g_{0}^{k}$, Lemmas 1.2.1 and 1.2.2, we see that there exists a positive constant $E_{2}$ such that $\left|\frac{u_{0}^{k+1}-u_{0}^{k}}{\Delta t}\right| \leq E_{2}$ holds for every $k$.

Remark 1.2.2. Since $g$ is twice continuously differentiable with respect to $x$ and $Q$, and $w$ is continuously differentiable with respect to $x$, in view of (1.2.4)-(1.2.5) and Remark 1.2.1 there exists a positive constant $\omega_{3}$ such that

$$
\begin{aligned}
\max _{1 \leq j \leq n}\left|D_{\Delta x}^{-}\left(g_{j}^{k}\right)\right|+\max _{1 \leq j \leq n}\left|m_{j}^{k}\right| \leq & \max _{(x, Q) \in \mathbb{D}}\left|g_{x}(x, Q)\right|+\max _{(x, Q) \in \mathbb{D}}\left|g_{Q}(x, Q)\right|(1-\alpha)\|w\|_{\infty}\left\|u^{k}\right\|_{\infty} \\
& +\max _{(x, Q) \in \mathbb{D}}|m(x, Q)| \\
\leq & \omega_{2}+\max _{(x, Q) \in \mathbb{D}}\left|g_{Q}(x, Q)\right|(1-\alpha)\|w\|_{\infty}\left\|u^{k}\right\|_{\infty} \leq \omega_{3} .
\end{aligned}
$$

Now set $\eta_{j}^{k}=D_{\Delta x}^{-}\left(u_{j}^{k}\right)$. We have the following result.

Lemma 1.2.5. Assume that $\triangle t$ is chosen to satisfy $\omega_{3} \triangle t<1$, then there exist positive constants $E_{3}$ and $E_{4}$ such that

$$
\begin{align*}
\sum_{j=2}^{n} & {\left[D_{\Delta x}^{-}\left(D_{\Delta x}^{-}\left(g_{j}^{k} u_{j}^{k+1}\right)\right)+D_{\Delta x}^{-}\left(m_{j}^{k} u_{j}^{k+1}\right)\right] \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x } \\
& +\left[D_{\Delta x}^{-}\left(g_{1}^{k} u_{1}^{k+1}\right)+m_{1}^{k} u_{1}^{k+1}\right] \operatorname{sgn}\left(\eta_{1}^{k+1}\right)  \tag{1.2.10}\\
\geq- & \left(\omega_{3}\left\|\eta^{k+1}\right\|_{1}+E_{3}\left\|\eta^{k}\right\|_{1}+E_{4}\right)
\end{align*}
$$

Proof. We first consider the terms $\sum_{j=2}^{n} D_{\Delta x}^{-}\left(D_{\Delta x}^{-}\left(g_{j}^{k} u_{j}^{k+1}\right)\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x$ and $D_{\Delta x}^{-}\left(g_{1}^{k} u_{1}^{k+1}\right) \operatorname{sgn}\left(\eta_{1}^{k+1}\right)$. Straightforward computations give

$$
\begin{aligned}
& \sum_{j=2}^{n} D_{\Delta x}^{-}\left(\frac{g_{j}^{k} u_{j}^{k+1}-g_{j-1}^{k} u_{j-1}^{k+1}}{\triangle x}\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x \\
= & \sum_{j=2}^{n} D_{\Delta x}^{-}\left(\frac{g_{j}^{k}-g_{j-1}^{k}}{\triangle x} u_{j-1}^{k+1}\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x+\sum_{j=2}^{n} D_{\Delta x}^{-}\left(g_{j}^{k} \frac{u_{j}^{k+1}-u_{j-1}^{k+1}}{\triangle x}\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x
\end{aligned}
$$

and

$$
D_{\Delta x}^{-}\left(g_{1}^{k} u_{1}^{k+1}\right) \operatorname{sgn}\left(\eta_{1}^{k+1}\right)=D_{\Delta x}^{-}\left(g_{1}^{k}\right) u_{0}^{k+1} \operatorname{sgn}\left(\eta_{1}^{k+1}\right)+g_{1}^{k}\left|\eta_{1}^{k+1}\right| .
$$

Furthermore,

$$
\begin{aligned}
& \sum_{j=2}^{n} D_{\Delta x}^{-}\left(g_{j}^{k} \frac{u_{j}^{k+1}-u_{j-1}^{k+1}}{\triangle x}\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x+g_{1}^{k}\left|\eta_{1}^{k+1}\right| \\
= & \sum_{j=2}^{n}\left|\eta_{j}^{k+1}\right| D_{\Delta x}^{-}\left(g_{j}^{k}\right) \triangle x+\sum_{j=2}^{n} g_{j-1}^{k} \frac{\eta_{j}^{k+1}-\eta_{j-1}^{k+1}}{\triangle x} \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x+g_{1}^{k}\left|\eta_{1}^{k+1}\right| \\
\geq & \sum_{j=2}^{n}\left|\eta_{j}^{k+1}\right| D_{\Delta x}^{-}\left(g_{j}^{k}\right) \triangle x+\sum_{j=2}^{n} g_{j-1}^{k}\left(\left|\eta_{j}^{k+1}\right|-\left|\eta_{j-1}^{k+1}\right|\right)+g_{1}^{k}\left|\eta_{1}^{k+1}\right| \\
= & \sum_{j=2}^{n}\left(\left|\eta_{j}^{k+1}\right| g_{j}^{k}-\left|\eta_{j-1}^{k+1}\right| g_{j-1}^{k}\right)+g_{1}^{k}\left|\eta_{1}^{k+1}\right|=g_{n}^{k}\left|\eta_{n}^{k+1}\right|=0 .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \sum_{j=2}^{n} D_{\Delta x}^{-}\left(D_{\Delta x}^{-}\left(g_{j}^{k} u_{j}^{k+1}\right)\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \Delta x+D_{\Delta x}^{-}\left(g_{1}^{k} u_{1}^{k+1}\right) \operatorname{sgn}\left(\eta_{1}^{k+1}\right) \\
\geq & \sum_{j=2}^{n} D_{\Delta x}^{-}\left(\frac{g_{j}^{k}-g_{j-1}^{k}}{\triangle x} u_{j-1}^{k+1}\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x+D_{\Delta x}^{-}\left(g_{1}^{k}\right) u_{0}^{k+1} \operatorname{sgn}\left(\eta_{1}^{k+1}\right) \\
= & \sum_{j=2}^{n} D_{\Delta x}^{-}\left(D_{\Delta x}^{-}\left(g_{j}^{k}\right)\right) u_{j-1}^{k+1} \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x+\sum_{j=1}^{n-1} D_{\Delta x}^{-}\left(g_{j}^{k}\right) D_{\Delta x}^{-}\left(u_{j}^{k+1}\right) \operatorname{sgn}\left(\eta_{j+1}^{k+1}\right) \triangle x \\
& +D_{\Delta x}^{-}\left(g_{1}^{k}\right) u_{0}^{k+1} \operatorname{sgn}\left(\eta_{1}^{k+1}\right) \\
= & I_{1}+I_{2}+I_{3} .
\end{aligned}
$$

Moreover, we find

$$
\begin{aligned}
D_{\Delta x}^{-}\left(D_{\Delta x}^{-}\left(g_{j}^{k}\right)\right)= & \frac{g_{x}\left(\bar{x}_{j}, Q_{j}^{k}\right)-g_{x}\left(\bar{x}_{j-1}, Q_{j-1}^{k}\right)}{\triangle x} \\
& +\frac{g_{Q}\left(x_{j-1}, \bar{Q}_{j}^{k}\right)\left(Q_{j}^{k}-Q_{j-1}^{k}\right)-g_{Q}\left(x_{j-2}, \bar{Q}_{j-1}^{k}\right)\left(Q_{j-1}^{k}-Q_{j-2}^{k}\right)}{(\triangle x)^{2}},
\end{aligned}
$$

where $\bar{x}_{j} \in\left(x_{j-1}, x_{j}\right), \bar{x}_{j-1} \in\left(x_{j-2}, x_{j-1}\right), \bar{Q}_{j}^{k}$ is between $Q_{j-1}^{k}$ and $Q_{j}^{k}$, and $\bar{Q}_{j-1}^{k}$ is between $Q_{j-2}^{k}$ and $Q_{j-1}^{k}$. Rewriting the right side of the above equation yields

$$
\frac{g_{x}\left(\bar{x}_{j}, Q_{j}^{k}\right)-g_{x}\left(\bar{x}_{j-1}, Q_{j-1}^{k}\right)}{\triangle x}=g_{x x}\left(\widetilde{\bar{x}}_{j}, Q_{j}^{k}\right) \frac{\bar{x}_{j}-\bar{x}_{j-1}}{\triangle x}+g_{x Q}\left(\bar{x}_{j-1}, \widetilde{Q}_{j}^{k}\right)(\alpha-1) w_{j} u_{j}^{k},
$$

and

$$
\begin{aligned}
& \frac{g_{Q}\left(x_{j-1}, \bar{Q}_{j}^{k}\right)\left(Q_{j}^{k}-Q_{j-1}^{k}\right)-g_{Q}\left(x_{j-2}, \bar{Q}_{j-1}^{k}\right)\left(Q_{j-1}^{k}-Q_{j-2}^{k}\right)}{(\triangle x)^{2}} \\
= & g_{Q x}\left(\tilde{x}_{j-1}, \bar{Q}_{j}^{k}\right)(\alpha-1) w_{j} u_{j}^{k}+g_{Q Q}\left(x_{j-2}, \widetilde{\bar{Q}}_{j}^{k}\right)(\alpha-1) w_{j-1} u_{j-1}^{k} \frac{\bar{Q}_{j}^{k}-\bar{Q}_{j-1}^{k}}{\triangle x} \\
& +g_{Q}\left(x_{j-2}, \bar{Q}_{j}^{k}\right)(\alpha-1) w^{\prime}\left(\hat{x}_{j}\right) u_{j}^{k}+g_{Q}\left(x_{j-2}, \bar{Q}_{j}^{k}\right)(\alpha-1) w\left(x_{j-1}\right) \eta_{j}^{k},
\end{aligned}
$$

where $\widetilde{\bar{x}}_{j} \in\left(\bar{x}_{j-1}, \bar{x}_{j}\right), \widetilde{Q}_{j}^{k}$ is between $Q_{j-1}^{k}$ and $Q_{j}^{k}, \tilde{x}_{j-1} \in\left(x_{j-2}, x_{j-1}\right), \hat{x}_{j} \in\left(x_{j-1}, x_{j}\right)$,
and $\widetilde{\bar{Q}}_{j}^{k}$ is between $\bar{Q}_{j-1}^{k}$ and $\bar{Q}_{j}^{k}$. For simplicity, we may let $I_{1}=J_{1}+J_{2}$, where

$$
J_{1}=\sum_{j=2}^{n} g_{Q}\left(x_{j-2}, \bar{Q}_{j}^{k}\right)(\alpha-1) w\left(x_{j-1}\right) u_{j-1}^{k+1} \eta_{j}^{k} \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x
$$

and $J_{2}$ contains the remaining terms in $I_{1}$.
We then consider the other term $\sum_{j=2}^{n} D_{\Delta x}^{-}\left(m_{j}^{k} u_{j}^{k+1}\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x$. Simple calculations yield

$$
\begin{aligned}
& \sum_{j=2}^{n} D_{\Delta x}^{-}\left(m_{j}^{k} u_{j}^{k+1}\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x \\
= & \sum_{j=2}^{n} D_{\Delta x}^{-}\left(m_{j}^{k}\right) u_{j-1}^{k+1} \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x+\sum_{j=2}^{n} m_{j}^{k} D_{\Delta x}^{-}\left(u_{j}^{k+1}\right) \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x \\
= & \sum_{j=2}^{n}\left[m_{x}\left(\tilde{x}_{j}, Q_{j}^{k}\right)+m_{Q}\left(x_{j}, \widetilde{Q}_{j}^{k}\right)(\alpha-1) w_{j} u_{j}^{k}\right] u_{j-1}^{k+1} \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x+\sum_{j=2}^{n} m_{j}^{k}\left|\eta_{j}^{k+1}\right| \triangle x \\
= & I_{4}+I_{5},
\end{aligned}
$$

where $\tilde{x}_{j} \in\left(x_{j-1}, x_{j}\right)$ and $\widetilde{Q}_{j}^{k}$ is between $Q_{j-1}^{k}$ and $Q_{j}^{k}$.
By Remark 1.2.2, $I_{2}+I_{5} \geq-\omega_{3}\left\|\eta^{k+1}\right\|_{1}$. Then by Lemmas 1.2.1 and 1.2.2, there exists a positive constant $E_{3}$ such that $J_{1} \geq-E_{3}\left\|\eta^{k}\right\|_{1}$. Furthermore, noticing the fact that

$$
\left|\frac{\bar{Q}_{j}^{k}-\bar{Q}_{j-1}^{k}}{\triangle x}\right| \leq\left|\frac{Q_{j}^{k}-Q_{j-1}^{k}}{\triangle x}\right|+\left|\frac{Q_{j-1}^{k}-Q_{j-2}^{k}}{\triangle x}\right| \leq 2(1-\alpha)\|w\|_{\infty}\left\|u^{k}\right\|_{\infty}
$$

we see that there is another positive constant $E_{4}$ such that

$$
J_{2}+I_{3}+I_{4}+m_{1} u_{1}^{k+1} \operatorname{sgn}\left(\eta_{1}^{k+1}\right) \geq-E_{4} .
$$

Thus we obtain (1.2.10).

With the above lemmas, we now can show that the approximation $u_{j}^{k}$ has bounded total variation. This bound plays an important role in establishing the subsequential convergence of the difference approximation (1.2.1) to a weak solution of (1.1.1).

Lemma 1.2.6. Assume that $\Delta t$ is chosen to satisfy $\omega_{3} \Delta t<1$. Then there exists $a$ positive constant $E_{5}$ such that $\left\|D_{\Delta x}^{-}\left(u^{k}\right)\right\|_{1} \leq E_{5}$.

Proof. Apply the operator $D_{\Delta x}^{-}$to the first equation of (1.2.1) to get

$$
\begin{equation*}
\frac{\eta_{j}^{k+1}-\eta_{j}^{k}}{\Delta t}+D_{\Delta x}^{-}\left(\frac{g_{j}^{k} u_{j}^{k+1}-g_{j-1}^{k} u_{j-1}^{k+1}}{\triangle x}\right)+D_{\Delta x}^{-}\left(m_{j}^{k} u_{j}^{k+1}\right)=0,2 \leq j \leq n . \tag{1.2.11}
\end{equation*}
$$

If $j=1$, the first equation of (1.2.1) takes the form

$$
\frac{u_{1}^{k+1}-u_{1}^{k}}{\triangle t}+\frac{g_{1}^{k} u_{1}^{k+1}-g_{0}^{k} u_{0}^{k+1}}{\triangle x}+m_{1}^{k} u_{1}^{k+1}=0
$$

On the other hand,

$$
\frac{\eta_{1}^{k+1}-\eta_{1}^{k}}{\Delta t}=\frac{1}{\triangle t}\left(\frac{u_{1}^{k+1}-u_{0}^{k+1}}{\triangle x}-\frac{u_{1}^{k}-u_{0}^{k}}{\triangle x}\right)=\frac{1}{\triangle x}\left(\frac{u_{1}^{k+1}-u_{1}^{k}}{\Delta t}-\frac{u_{0}^{k+1}-u_{0}^{k}}{\Delta t}\right)
$$

Hence,

$$
\begin{equation*}
\frac{\eta_{1}^{k+1}-\eta_{1}^{k}}{\Delta t}=-\frac{1}{\triangle x}\left(\frac{u_{0}^{k+1}-u_{0}^{k}}{\Delta t}+D_{\Delta x}^{-}\left(g_{1}^{k} u_{1}^{k+1}\right)+m_{1}^{k} u_{1}^{k+1}\right) . \tag{1.2.12}
\end{equation*}
$$

Multiplying (1.2.11) by $\operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x$ and noticing that $-\eta_{j}^{k} \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \geq-\left|\eta_{j}^{k}\right|$, we have

$$
\frac{\left|\eta_{j}^{k+1}\right|-\left|\eta_{j}^{k}\right|}{\triangle t} \triangle x+\left[D_{\Delta x}^{-}\left(\frac{g_{j}^{k} u_{j}^{k+1}-g_{j-1}^{k} u_{j-1}^{k+1}}{\triangle x}\right)+D_{\Delta x}^{-}\left(m_{j}^{k} u_{j}^{k+1}\right)\right] \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \triangle x \leq 0
$$

for $2 \leq j \leq n$. Similarly, multiplying (1.2.12) by $\operatorname{sgn}\left(\eta_{1}^{k+1}\right) \triangle x$, we have

$$
\frac{\left|\eta_{1}^{k+1}\right|-\left|\eta_{1}^{k}\right|}{\Delta t} \Delta x+\left[\frac{u_{0}^{k+1}-u_{0}^{k}}{\triangle t}+D_{\Delta x}^{-}\left(g_{1}^{k} u_{1}^{k+1}\right)+m_{1}^{k} u_{1}^{k+1}\right] \operatorname{sgn}\left(\eta_{1}^{k+1}\right) \leq 0
$$

Summing over the indices $j=1,2, \ldots, n$, we obtain

$$
\begin{aligned}
& \frac{\left\|\eta^{k+1}\right\|_{1}-\left\|\eta^{k}\right\|_{1}}{\Delta t}+\sum_{j=2}^{n}\left[D_{\Delta x}^{-}\left(D_{\Delta x}^{-}\left(g_{j}^{k} u_{j}^{k+1}\right)\right)+D_{\Delta x}^{-}\left(m_{j}^{k} u_{j}^{k+1}\right)\right] \operatorname{sgn}\left(\eta_{j}^{k+1}\right) \Delta x \\
& \quad+\left[D_{\Delta x}^{-}\left(g_{1}^{k} u_{1}^{k+1}\right)+m_{1}^{k} u_{1}^{k+1}\right] \operatorname{sgn}\left(\eta_{1}^{k+1}\right)-\left|\frac{u_{0}^{k+1}-u_{0}^{k}}{\Delta t}\right| \leq 0
\end{aligned}
$$

Then by Lemmas 1.2.4 and 1.2.5, we have

$$
\frac{\left\|\eta^{k+1}\right\|_{1}-\left\|\eta^{k}\right\|_{1}}{\triangle t} \leq \omega_{3}\left\|\eta^{k+1}\right\|_{1}+E_{3}\left\|\eta^{k}\right\|_{1}+E_{2}+E_{4}
$$

The above inequality leads to the desired result.

The next result shows that the difference approximation satisfies a Lipschitz-type condition in $t$.

Lemma 1.2.7. Assume that $\Delta t$ is chosen to satisfy $\omega_{3} \triangle t<1$. Then there exists a positive constant $E_{6}$ such that for any $q>p$, we have

$$
\sum_{j=1}^{n}\left|\frac{u_{j}^{q}-u_{j}^{p}}{\triangle t}\right| \triangle x \leq E_{6}(q-p)
$$

Proof. Using the first equation of (1.2.1), we obtain

$$
\begin{aligned}
\sum_{j=1}^{n}\left|\frac{u_{j}^{k+1}-u_{j}^{k}}{\triangle t} \triangle x\right| & =\sum_{j=1}^{n}\left|\left(\frac{g_{j}^{k} u_{j}^{k+1}-g_{j-1}^{k} u_{j-1}^{k+1}}{\triangle x}+m_{j}^{k} u_{j}^{k+1}\right) \triangle x\right| \\
& =\sum_{j=1}^{n}\left|\left[\left(\frac{g_{j}^{k}-g_{j-1}^{k}}{\triangle x}+m_{j}^{k}\right) u_{j}^{k+1}+g_{j-1}^{k} \frac{u_{j}^{k+1}-u_{j-1}^{k+1}}{\triangle x}\right] \triangle x\right|
\end{aligned}
$$

Hence, there exists a positive constant $E_{6}$ such that

$$
\begin{aligned}
\sum_{j=1}^{n}\left|\frac{u_{j}^{k+1}-u_{j}^{k}}{\triangle t}\right| \triangle x & \leq \sum_{j=1}^{n}\left|D_{\Delta x}^{-}\left(g_{j}^{k}\right)+m_{j}^{k}\right| u_{j}^{k+1} \triangle x+\sum_{j=1}^{n} g_{j-1}^{k}\left|\eta_{j}^{k+1}\right| \triangle x \\
& \leq \omega_{3}\left\|u^{k+1}\right\|_{1}+\max _{(x, Q) \in \mathbb{D}} g(x, Q)\left\|\eta^{k+1}\right\|_{1} \leq E_{6}
\end{aligned}
$$

Thus,

$$
\sum_{j=1}^{n}\left|\frac{u_{j}^{q}-u_{j}^{p}}{\Delta t}\right| \Delta x \leq \sum_{k=p}^{q} \sum_{j=1}^{n}\left|\frac{u_{j}^{k+1}-u_{j}^{k}}{\Delta t}\right| \triangle x \leq E_{6}(q-p)
$$

Following [10] we define a family of functions $\left\{U_{\Delta x, \Delta t}\right\}$ by

$$
U_{\Delta x, \Delta t}(x, t)=u_{j}^{k}
$$

for $x \in\left[x_{j-1}, x_{j}\right), t \in\left[t_{k-1}, t_{k}\right), j=1, \ldots, n$ and $k=1, \ldots, l$. Then, by Lemmas 1.2.1, 1.2.2, 1.2.6 and 1.2.7, the set of functions $\left\{U_{\Delta x, \Delta t}\right\}$ is compact in the topology of $\mathcal{L}^{1}((0, L) \times(0, T))$, and hence following the proof of Lemma 16.7 (p.276) in [10] we have the following lemma.

Lemma 1.2.8. There exists a sequence $\left\{U_{\triangle x_{i}, \Delta t_{i}}\right\} \subset\left\{U_{\Delta x, \Delta t}\right\}$ which converges to $a$ $B V([0, L] \times[0, T])$ function $u(x, t)$ in the sense that for all $t>0$

$$
\int_{0}^{L}\left|U_{\Delta x_{i}, \Delta t_{i}}-u(x, t)\right| d x \rightarrow 0
$$

and

$$
\int_{0}^{T} \int_{0}^{L}\left|U_{\triangle x_{i}, \Delta t_{i}}-u(x, t)\right| d x d t \rightarrow 0
$$

as $i \rightarrow \infty$. Furthermore, there exists a positive constant $E_{7}$ (dependent on $\left\|u^{0}\right\|_{B V[0, L]}$ and $\left.\|C\|_{\mathcal{C}^{1}[0, T]}\right)$ such that the function $u$ satisfies

$$
\|u\|_{B V([0, L] \times[0, T])} \leq E_{7} .
$$

The next theorem shows that the limit function $u(x, t)$ constructed via our difference scheme is a weak solution of problem (1.1.1).

Theorem 1.2.9. The limit function $u(x, t)$ defined in Lemma 1.2.8 is a weak solution of (1.1.1) and satisfies

$$
Q(x, t) \leq\|w\|_{\infty}\|u(\cdot, t)\|_{1} \leq\|w\|_{\infty}\left(e^{\omega_{1} T}\left\|u^{0}\right\|_{1}+\int_{0}^{T} e^{\omega_{1}(T-s)} C(s) d s\right)
$$

and

$$
\|u\|_{\mathcal{L}^{\infty}((0, L) \times(0, T))} \leq \max \left\{e^{\omega_{2} T}\left\|u^{0}\right\|_{\infty}, \frac{\|C\|_{\mathcal{C}[0, T]}+\omega_{1}\|u\|_{\mathcal{L}^{\infty}\left((0, T) ; \mathcal{L}^{1}(0, L)\right)}}{\mu}\right\},
$$

where $\mu \leq g(0, Q)$ for $Q \in\left[0, Q_{\max }\right]$.

Proof. Let $\varphi \in \mathcal{C}^{1}((0, L) \times(0, T))$ and denote the finite difference approximations $\varphi\left(x_{j}, t_{k}\right)$ by $\varphi_{j}^{k}$. Multiplying the difference scheme by $\varphi_{j}^{k+1}$, we get

$$
\begin{aligned}
& \frac{u_{j}^{k+1} \varphi_{j}^{k+1}-u_{j}^{k} \varphi_{j}^{k}}{\Delta t}-u_{j}^{k} \frac{\varphi_{j}^{k+1}-\varphi_{j}^{k}}{\Delta t}+\frac{g_{j}^{k} u_{j}^{k+1} \varphi_{j}^{k+1}-g_{j-1}^{k} u_{j-1}^{k+1} \varphi_{j-1}^{k+1}}{\Delta x} \\
& -g_{j-1}^{k} u_{j-1}^{k+1} \frac{\varphi_{j}^{k+1}-\varphi_{j-1}^{k+1}}{\Delta x}+m_{j}^{k} u_{j}^{k+1} \varphi_{j}^{k+1}=0 .
\end{aligned}
$$

Multiplying the above equation by $\Delta x \Delta t$, and summing over $k=0,1,2, \ldots, i-1$ and $j=1,2, \ldots, n$, we obtain

$$
\begin{aligned}
& \sum_{j=1}^{n}\left(u_{j}^{i} \varphi_{j}^{i}-u_{j}^{0} \varphi_{j}^{0}\right) \Delta x \\
= & \sum_{k=0}^{i-1} \sum_{j=1}^{n}\left(u_{j}^{k} \frac{\varphi_{j}^{k+1}-\varphi_{j}^{k}}{\Delta t}+g_{j-1}^{k} u_{j-1}^{k+1} \frac{\varphi_{j}^{k+1}-\varphi_{j-1}^{k+1}}{\Delta x}-m_{j}^{k} u_{j}^{k+1} \varphi_{j}^{k+1}\right) \Delta x \Delta t \\
& +\sum_{k=0}^{i-1} \varphi_{0}^{k+1}\left(C^{k}+\sum_{j=1}^{n} \beta_{j}^{k} u_{j}^{k} \triangle x\right) \Delta t .
\end{aligned}
$$

Passing to the limit, we find that $u(x, t)$ satisfies the weak solution condition. Taking the limit in the bounds obtained in Lemmas 1.2.1 and 1.2.2, we get the above bounds on $Q(x, t)$ and $\|u\|_{\mathcal{L}^{\infty}((0, L) \times(0, T))}$, respectively.

The following theorem guarantees the continuous dependence of the solution $\left\{u_{j}^{k}\right\}$ of (1.2.1) with respect to the initial condition $u^{0}$.

Theorem 1.2.10. Let $\left\{u_{j}^{k}\right\}$ and $\left\{\hat{u}_{j}^{k}\right\}$ be the solutions of our scheme (1.2.1) corresponding to the initial conditions $u_{j}^{0}$ and $\hat{u}_{j}^{0}$ and the boundary conditions $C^{k}$ and $\widehat{C}^{k}$, respectively. Then, there exists a positive constant $\delta$ such that

$$
\begin{equation*}
\left\|u^{k+1}-\hat{u}^{k+1}\right\|_{1} \leq(1+\delta \triangle t)\left\|u^{k}-\hat{u}^{k}\right\|_{1}+\triangle t\left|C^{k}-\widehat{C}^{k}\right| . \tag{1.2.13}
\end{equation*}
$$

Proof. Since $\left\{u_{j}^{k}\right\}$ and $\left\{\hat{u}_{j}^{k}\right\}$ satisfy

$$
\begin{aligned}
& \frac{u_{j}^{k+1}-u_{j}^{k}}{\Delta t}+D_{\Delta x}^{-}\left(g_{j}^{k} u_{j}^{k+1}\right)+m_{j}^{k} u_{j}^{k+1}=0, \\
& \frac{g_{0}^{k} u_{0}^{k+1}=C^{k}+\sum_{j=1}^{n} \beta_{j}^{k} u_{j}^{k} \Delta x,}{} \begin{array}{l}
\hat{u}_{j}^{k+1}-\hat{u}_{j}^{k} \\
\triangle t
\end{array}+D_{\Delta x}^{-}\left(\hat{g}_{j}^{k} \hat{u}_{j}^{k+1}\right)+\hat{m}_{j}^{k} \hat{u}_{j}^{k+1}=0, \quad \hat{g}_{0}^{k} \hat{u}_{0}^{k+1}=\widehat{C}^{k}+\sum_{j=1}^{n} \hat{\beta}_{j}^{k} \hat{u}_{j}^{k} \triangle x,
\end{aligned}
$$

respectively (here $\hat{g}_{j}^{k}=g\left(x_{j}, \widehat{Q}_{j}^{k}\right)$, and similar notation is used for $\hat{m}_{j}^{k}$ and $\hat{\beta}_{j}^{k}$ ), letting $v_{j}^{k}=u_{j}^{k}-\hat{u}_{j}^{k}$, we get

$$
\begin{gather*}
\frac{v_{j}^{k+1}-v_{j}^{k}}{\Delta t}+D_{\Delta x}^{-}\left(g_{j}^{k} u_{j}^{k+1}-\hat{g}_{j}^{k} \hat{u}_{j}^{k+1}\right)+m_{j}^{k} u_{j}^{k+1}-\hat{m}_{j}^{k} \hat{u}_{j}^{k+1}=0, \quad 1 \leq j \leq n  \tag{1.2.14}\\
g_{0}^{k} u_{0}^{k+1}-\hat{g}_{0}^{k} \hat{u}_{0}^{k+1}=\left(C^{k}-\widehat{C}^{k}\right)+\sum_{j=1}^{n} \beta_{j}^{k} u_{j}^{k} \triangle x-\sum_{j=1}^{n} \hat{\beta}_{j}^{k} \hat{u}_{j}^{k} \triangle x \tag{1.2.15}
\end{gather*}
$$

Multiplying (1.2.14) by $\operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x$, noticing $-v_{j}^{k} \operatorname{sgn}\left(v_{j}^{k+1}\right) \geq-\left|v_{j}^{k}\right|$ and summing over $j=1,2, \ldots, n$, we find

$$
\begin{equation*}
\frac{\left\|v^{k+1}\right\|_{1}-\left\|v^{k}\right\|_{1}}{\triangle t} \leq-\sum_{j=1}^{n}\left[D_{\Delta x}^{-}\left(g_{j}^{k} u_{j}^{k+1}-\hat{g}_{j}^{k} \hat{u}_{j}^{k+1}\right)+m_{j}^{k} u_{j}^{k+1}-\hat{m}_{j}^{k} \hat{u}_{j}^{k+1}\right] \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x \tag{1.2.16}
\end{equation*}
$$

Note that

$$
\begin{aligned}
\sum_{j=1}^{n} D_{\Delta x}^{-}\left(g_{j}^{k} v_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \Delta x & =\sum_{j=1}^{n}\left[v_{j}^{k+1} D_{\Delta x}^{-}\left(g_{j}^{k}\right)+g_{j-1}^{k} D_{\Delta x}^{-}\left(v_{j}^{k+1}\right)\right] \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x \\
& \geq \sum_{j=1}^{n}\left|v_{j}^{k+1}\right|\left(g_{j}^{k}-g_{j-1}^{k}\right)+\sum_{j=1}^{n} g_{j-1}^{k}\left(\left|v_{j}^{k+1}\right|-\left|v_{j-1}^{k+1}\right|\right) \\
& =\left|v_{n}^{k+1}\right| g_{n}^{k}-\left|v_{0}^{k+1}\right| g_{0}^{k}=-\left|v_{0}^{k+1}\right| g_{0}^{k} .
\end{aligned}
$$

Hence,

$$
\begin{align*}
& \sum_{j=1}^{n} D_{\Delta x}^{-}\left(g_{j}^{k} u_{j}^{k+1}-\hat{g}_{j}^{k} \hat{u}_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x \\
= & \sum_{j=1}^{n} D_{\Delta x}^{-}\left(g_{j}^{k} v_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x+\sum_{j=1}^{n} D_{\Delta x}^{-}\left(\left(g_{j}^{k}-\hat{g}_{j}^{k}\right) \hat{u}_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \Delta x  \tag{1.2.17}\\
\geq & \sum_{j=1}^{n} D_{\Delta x}^{-}\left(\left(g_{j}^{k}-\hat{g}_{j}^{k}\right) \hat{u}_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x-\left|v_{0}^{k+1}\right| g_{0}^{k} .
\end{align*}
$$

Furthermore, we have

$$
\begin{align*}
& \sum_{j=1}^{n}\left(m_{j}^{k} u_{j}^{k+1}-\hat{m}_{j}^{k} \hat{u}_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x \\
= & \sum_{j=1}^{n} m_{j}^{k}\left|v_{j}^{k+1}\right| \triangle x+\sum_{j=1}^{n}\left(m_{j}^{k}-\hat{m}_{j}^{k}\right) \hat{u}_{j}^{k+1} \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x  \tag{1.2.18}\\
\geq & \sum_{j=1}^{n}\left(m_{j}^{k}-\hat{m}_{j}^{k}\right) \hat{u}_{j}^{k+1} \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x .
\end{align*}
$$

By (1.2.16)-(1.2.18), we obtain

$$
\begin{align*}
\frac{\left\|v^{k+1}\right\|_{1}-\left\|v^{k}\right\|_{1}}{\triangle t} \leq & -\sum_{j=1}^{n} D_{\Delta x}^{-}\left(\left(g_{j}^{k}-\hat{g}_{j}^{k}\right) \hat{u}_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x  \tag{1.2.19}\\
& -\sum_{j=1}^{n}\left(m_{j}^{k}-\hat{m}_{j}^{k}\right) \hat{u}_{j}^{k+1} \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x+\left|v_{0}^{k+1}\right| g_{0}^{k}
\end{align*}
$$

On the other hand, upon manipulation we have

$$
\begin{aligned}
& D_{\Delta x}^{-}\left(g_{j}^{k}-\hat{g}_{j}^{k}\right) \Delta x \\
= & {\left[g\left(x_{j}, Q_{j}^{k}\right)-g\left(x_{j-1}, Q_{j-1}^{k}\right)\right]-\left[g\left(x_{j}, \widehat{Q}_{j}^{k}\right)-g\left(x_{j-1}, \widehat{Q}_{j-1}^{k}\right)\right] } \\
= & {\left[\int_{0}^{1} g_{x}\left(r x_{j}+(1-r) x_{j-1}, r Q_{j}^{k}+(1-r) Q_{j-1}^{k}\right) d r \triangle x\right.} \\
& \left.-\int_{0}^{1} g_{x}\left(r x_{j}+(1-r) x_{j-1}, r \widehat{Q}_{j}^{k}+(1-r) \widehat{Q}_{j-1}^{k}\right) d r \triangle x\right] \\
& +\left[\int_{0}^{1} g_{Q}\left(r x_{j}+(1-r) x_{j-1}, r Q_{j}^{k}+(1-r) Q_{j-1}^{k}\right) d r\left(Q_{j}^{k}-Q_{j-1}^{k}\right)\right. \\
& \left.-\int_{0}^{1} g_{Q}\left(r x_{j}+(1-r) x_{j-1}, r \widehat{Q}_{j}^{k}+(1-r) \widehat{Q}_{j-1}^{k}\right) d r\left(\widehat{Q}_{j}^{k}-\widehat{Q}_{j-1}^{k}\right)\right] \\
= & \int_{0}^{1} g_{x Q}\left(\bar{x}_{j}, \bar{Q}_{j}\right)\left[r\left(Q_{j}^{k}-\widehat{Q}_{j}^{k}\right)+(1-r)\left(Q_{j-1}^{k}-\widehat{Q}_{j-1}^{k}\right)\right] d r \triangle x \\
& +\int_{0}^{1} g_{Q Q}\left(\bar{x}_{j}, \widetilde{Q}_{j}\right)\left[r\left(Q_{j}^{k}-\widehat{Q}_{j}^{k}\right)+(1-r)\left(Q_{j-1}^{k}-\widehat{Q}_{j-1}^{k}\right)\right] d r(\alpha-1) w_{j} u_{j}^{k} \triangle x \\
& +\int_{0}^{1} g_{Q}\left(\bar{x}_{j}, r \widehat{Q}_{j}^{k}+(1-r) \widehat{Q}_{j-1}^{k}\right) d r(\alpha-1) w_{j} v_{j}^{k} \triangle x,
\end{aligned}
$$

where $\bar{x}_{j}=r x_{j}+(1-r) x_{j-1}, \bar{Q}_{j}$ and $\widetilde{Q}_{j}$ are both between $r Q_{j}^{k}+(1-r) Q_{j-1}^{k}$ and $r \widehat{Q}_{j}^{k}+(1-r) \widehat{Q}_{j-1}^{k}$. Clearly,

$$
\begin{aligned}
& \sum_{j=1}^{n} D_{\Delta x}^{-}\left(\left(g_{j}^{k}-\hat{g}_{j}^{k}\right) \hat{u}_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x \\
= & \sum_{j=1}^{n} \hat{u}_{j-1}^{k+1} D_{\Delta x}^{-}\left(g_{j}^{k}-\hat{g}_{j}^{k}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x \\
& +\sum_{j=1}^{n} g_{Q}\left(x_{j}, \bar{Q}\right)\left(Q_{j}^{k}-\widehat{Q}_{j}^{k}\right) D_{\Delta x}^{-}\left(\hat{u}_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x
\end{aligned}
$$

where $\bar{Q}$ is between $Q_{j}^{k}$ and $\widehat{Q}_{j}^{k}$. Hence, there exist positive constants $c_{1}$ and $c_{2}$ such that

$$
\begin{align*}
& -\sum_{j=1}^{n} D_{\Delta x}^{-}\left(\left(g_{j}^{k}-\hat{g}_{j}^{k}\right) \hat{u}_{j}^{k+1}\right) \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x \\
\leq & \sum_{j=1}^{n} \hat{u}_{j-1}^{k+1}\left|D_{\Delta x}^{-}\left(g_{j}^{k}-\hat{g}_{j}^{k}\right)\right| \triangle x+\max _{(x, Q) \in \mathcal{D}}\left|g_{Q}(x, Q)\right| \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right|\left\|D_{\Delta x}^{-}\left(\hat{u}^{k+1}\right)\right\|_{1} \\
\leq & (1-\alpha)\|w\|_{\infty}\left\|\hat{u}^{k+1}\right\|_{\infty} \max _{(x, Q) \in \mathcal{D}}\left|g_{Q}(x, Q)\right|\left\|v^{k}\right\|_{1} \\
& +\left\|\hat{u}^{k+1}\right\|_{1} \max _{(x, Q) \in \mathcal{D}}\left|g_{x Q}(x, Q)\right| \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right| \\
& +(1-\alpha)\|w\|_{\infty}\left\|u^{k}\right\|_{\infty}\left\|\hat{u}^{k+1}\right\|_{1} \max _{(x, Q) \in \mathcal{D}}\left|g_{Q Q}(x, Q)\right| \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right| \\
& +\max _{(x, Q) \in \mathcal{D}}\left|g_{Q}(x, Q)\right|\left\|D_{\Delta x}^{-}\left(\hat{u}^{k+1}\right)\right\|_{1} \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right| \\
\leq & c_{1}\left\|v^{k}\right\|_{1}+c_{2} \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right|, \tag{1.2.20}
\end{align*}
$$

where $\mathcal{D}=\left\{(x, Q) \mid(x, Q) \in[0, L] \times\left[0, \bar{Q}_{\max }\right]\right.$, with $\left.\bar{Q}_{\max }=\max \left\{Q_{\max }, \widehat{Q}_{\max }\right\}\right\}$. Furthermore, by (1.2.15) we have

$$
g_{0}^{k} v_{0}^{k+1}+\left(g_{0}^{k}-\hat{g}_{0}^{k}\right) \hat{u}_{0}^{k+1}=\left(C^{k}-\widehat{C}^{k}\right)+\sum_{j=1}^{n} \beta_{j}^{k} v_{j}^{k} \triangle x+\sum_{j=1}^{n}\left(\beta_{j}^{k}-\hat{\beta}_{j}^{k}\right) \hat{u}_{j}^{k} \triangle x
$$

that is,

$$
\begin{aligned}
g_{0}^{k} v_{0}^{k+1}= & -g_{Q}\left(0, \bar{Q}_{0}\right)\left(Q_{0}^{k}-\widehat{Q}_{0}^{k}\right) \hat{u}_{0}^{k+1}+\sum_{j=1}^{n} \beta_{j}^{k} v_{j}^{k} \triangle x \\
& +\sum_{j=1}^{n} \beta_{Q}\left(x_{j}, \bar{Q}_{j}\right)\left(Q_{j}^{k}-\widehat{Q}_{j}^{k}\right) \hat{u}_{j}^{k} \triangle x+\left(C^{k}-\widehat{C}^{k}\right)
\end{aligned}
$$

where $\bar{Q}_{j}$ is between $Q_{j}^{k}$ and $\widehat{Q}_{j}^{k}, j=0,1, \ldots, n$. Hence, there exists a positive constant
$c_{3}$ such that

$$
\begin{align*}
g_{0}^{k}\left|v_{0}^{k+1}\right| \leq & \max _{(x, Q) \in \mathcal{D}}\left|g_{Q}(x, Q)\right| \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right|\left\|\hat{u}^{k+1}\right\|_{\infty}+\omega_{1}\left\|v^{k}\right\|_{1} \\
& +\max _{(x, Q) \in \mathcal{D}}\left|\beta_{Q}(x, Q)\right| \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right|\left\|\hat{u}^{k}\right\|_{1}+\left|C^{k}-\widehat{C}^{k}\right|  \tag{1.2.21}\\
\leq & \omega_{1}\left\|v^{k}\right\|_{1}+c_{3} \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right|+\left|C^{k}-\widehat{C}^{k}\right| .
\end{align*}
$$

On the other hand, we have

$$
\sum_{j=1}^{n}\left(m_{j}^{k}-\hat{m}_{j}^{k}\right) \hat{u}_{j}^{k+1} \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x=\sum_{j=1}^{n} m_{Q}\left(x_{j}, \bar{Q}_{j}\right)\left(Q_{j}^{k}-\widehat{Q}_{j}^{k}\right) \hat{u}_{j}^{k+1} \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x
$$

where $\bar{Q}_{j}$ is between $Q_{j}^{k}$ and $\widehat{Q}_{j}^{k}$. Hence, there exists a positive constant $c_{4}$ such that

$$
\begin{align*}
& -\sum_{j=1}^{n}\left(m_{j}^{k}-\hat{m}_{j}^{k}\right) \hat{u}_{j}^{k+1} \operatorname{sgn}\left(v_{j}^{k+1}\right) \triangle x  \tag{1.2.22}\\
& \quad \leq \max _{(x, Q) \in \mathcal{D}}\left|m_{Q}(x, Q)\right| \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right|\left\|\hat{u}^{k+1}\right\|_{1} \leq c_{4} \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right|
\end{align*}
$$

Thus, by (1.2.19)-(1.2.22) we find

$$
\begin{aligned}
\frac{\left\|v^{k+1}\right\|_{1}-\left\|v^{k}\right\|_{1}}{\Delta t} & \leq\left(c_{1}+\omega_{1}\right)\left\|v^{k}\right\|_{1}+\left|C^{k}-\widehat{C}^{k}\right|+\left(c_{2}+c_{3}+c_{4}\right) \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right| \\
& \leq\left[\left(c_{1}+\omega_{1}\right)+\left(c_{2}+c_{3}+c_{4}\right)\|w\|_{\infty}\right]\left\|v^{k}\right\|_{1}+\left|C^{k}-\widehat{C}^{k}\right|,
\end{aligned}
$$

since $0 \leq \alpha<1$ and

$$
Q_{j}^{k}-\widehat{Q}_{j}^{k}=\alpha \sum_{i=1}^{j} w_{i} v_{i}^{k} \triangle x+\sum_{i=j+1}^{n} w_{i} v_{i}^{k} \triangle x
$$

Choose a constant $\delta \geq\left(c_{1}+\omega_{1}\right)+\left(c_{2}+c_{3}+c_{4}\right)\|w\|_{\infty}$. We then have

$$
\frac{\left\|v^{k+1}\right\|_{1}-\left\|v^{k}\right\|_{1}}{\triangle t} \leq \delta\left\|v^{k}\right\|_{1}+\left|C^{k}-\widehat{C}^{k}\right|
$$

which yields (1.2.13).

Next, we prove that the $B V$ solution defined in Lemma 1.2.8 and Theorem 1.2.9 is unique.

Theorem 1.2.11. Suppose that $u$ and $\hat{u}$ are bounded variation weak solutions of problem (1.1.1) corresponding to the initial conditions $u_{0}$ and $\hat{u}_{0}$ and the boundary conditions $C$ and $\widehat{C}$, respectively, then there exist positive constants $\lambda$ and $\gamma$ such that

$$
\|u(\cdot, t)-\hat{u}(\cdot, t)\|_{1} \leq \lambda e^{\gamma t}\left(\|u(\cdot, 0)-\hat{u}(\cdot, 0)\|_{1}+\|C-\widehat{C}\|_{\mathcal{L}^{1}(0, T)}\right)
$$

Proof. Assume that $Q$ and $B$ are given Lipschitz continuous functions and consider the following initial-boundary value problem:

$$
\begin{array}{ll}
u_{t}+(g(x, Q(x, t)) u)_{x}+m(x, Q(x, t)) u=0, & (x, t) \in(0, L] \times(0, T] \\
g(0, Q(0, t)) u(0, t)=B(t), & t \in(0, T]  \tag{1.2.23}\\
u(x, 0)=u^{0}(x), & x \in[0, L] .
\end{array}
$$

Since (1.2.23) is a linear problem with a local boundary condition, it has a unique weak solution. In fact, a weak solution can be defined as a limit of the finite difference approximation with the given numbers $Q_{j}^{k}=Q\left(x_{j}, t_{k}\right)$ and $B^{k}=B\left(t_{k}\right)$, and the uniqueness can be established by using a similar technique as in ([10], p. 282). In addition, from the proof of Theorem 1.2.10, we can see that if $\left\{u_{j}^{k}\right\}$ and $\left\{\hat{u}_{j}^{k}\right\}$ are solutions of the difference scheme corresponding to given functions $\left(Q_{j}^{k}, B^{k}\right)$ and $\left(\widehat{Q}_{j}^{k}, \widehat{B}^{k}\right)$, respectively, we have

$$
\frac{\left\|v^{k+1}\right\|_{1}-\left\|v^{k}\right\|_{1}}{\triangle t} \leq c_{1}\left\|v^{k}\right\|_{1}+\left(c_{2}+c_{3}+c_{4}\right) \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right|+\left|B^{k}-\widehat{B}^{k}\right|
$$

Let $\rho=c_{2}+c_{3}+c_{4}$, then we have

$$
\left\|v^{k+1}\right\|_{1} \leq\left(1+c_{1} \triangle t\right)\left\|v^{k}\right\|_{1}+\rho \triangle t \max _{j}\left|Q_{j}^{k}-\widehat{Q}_{j}^{k}\right|+\left|B^{k}-\widehat{B}^{k}\right| \triangle t
$$

Equivalently,

$$
\begin{aligned}
\left\|v^{k}\right\|_{1} \leq & \left(1+c_{1} \triangle t\right)^{k}\left\|v^{0}\right\|_{1}+\sum_{i=0}^{k-1}\left(1+c_{1} \triangle t\right)^{i}\left[\left|B^{k-1-i}-\widehat{B}^{k-1-i}\right| \Delta t\right. \\
& \left.+\rho \triangle t \max _{j}\left|Q_{j}^{k-1-i}-\widehat{Q}_{j}^{k-1-i}\right|\right]
\end{aligned}
$$

Hence,

$$
\begin{equation*}
\left\|v^{k}\right\|_{1} \leq\left(1+c_{1} \triangle t\right)^{k}\left[\left\|v^{0}\right\|_{1}+\sum_{i=0}^{k-1}\left(\rho \triangle t \max _{j}\left|Q_{j}^{i}-\widehat{Q}_{j}^{i}\right|+\left|B^{i}-\widehat{B}^{i}\right| \triangle t\right)\right] \tag{1.2.24}
\end{equation*}
$$

From Theorem 1.2.9, we know that $\left\{U_{\Delta x, \Delta t}\right\}$ and $\left\{\widehat{U}_{\Delta x, \Delta t}\right\}$ converge to $u(x, t)$ and $\hat{u}(x, t)$ strongly in $\mathcal{C}\left([0, T] ; \mathcal{L}^{1}(0, L)\right)$, respectively, where $u(x, t)$ and $\hat{u}(x, t)$ are the unique solutions of (1.2.23) with the given functions $(Q(x, t), B(t))$ and $(\widehat{Q}(x, t), \widehat{B}(t))$, respectively. Letting $v(t)=u(\cdot, t)-\hat{u}(\cdot, t)$ and taking the limit of (1.2.24), we obtain

$$
\begin{equation*}
\|v(t)\|_{1} \leq e^{c_{1} T}\left[\|v(0)\|_{1}+\int_{0}^{t}\left(\rho \sup _{x \in[0, L]}|Q(x, s)-\widehat{Q}(x, s)|+|B(s)-\widehat{B}(s)|\right) d s\right] \tag{1.2.25}
\end{equation*}
$$

By virtue of Lemma 1.2.8, the corresponding solutions of (1.2.23) satisfy

$$
\begin{aligned}
& Q(x, t)=\alpha \int_{0}^{x} w(\xi) u(\xi, t) d \xi+\int_{x}^{L} w(\xi) u(\xi, t) d \xi \\
& B(t)=C(t)+\int_{0}^{L} \beta(x, Q(x, t)) u(x, t) d x \\
& \widehat{Q}(x, t)=\alpha \int_{0}^{x} w(\xi) \hat{u}(\xi, t) d \xi+\int_{x}^{L} w(\xi) \hat{u}(\xi, t) d \xi \\
& \widehat{B}(t)=\widehat{C}(t)+\int_{0}^{L} \beta(x, \widehat{Q}(x, t)) \hat{u}(x, t) d x
\end{aligned}
$$

Thus, we get

$$
\sup _{x \in[0, L]}|Q(x, s)-\widehat{Q}(x, s)| \leq \int_{0}^{L} w(\xi)|u(\xi, s)-\hat{u}(\xi, s)| d \xi \leq\|w\|_{\infty}\|v(s)\|_{1}
$$

and

$$
\begin{aligned}
& |B(s)-\widehat{B}(s)| \\
\leq & |C(s)-\widehat{C}(s)|+\int_{0}^{L} \beta(x, Q)|u(x, s)-\hat{u}(x, s)| d x+\int_{0}^{L}\left|\beta_{Q}(x, \widetilde{Q})\right||Q-\widehat{Q}| \hat{u}(x, s) d x \\
\leq & |C(s)-\widehat{C}(s)|+\omega_{1}\|v(s)\|_{1}+L\|w\|_{\infty} \max _{(x, Q) \in \mathcal{D}}\left|\beta_{Q}(x, Q)\right|\|\hat{u}\|_{\mathcal{L}^{\infty}((0, L) \times(0, T))}\|v(s)\|_{1} .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \int_{0}^{t}\left(\rho \sup _{x \in[0, L]}|Q(x, s)-\widehat{Q}(x, s)|+|B(s)-\widehat{B}(s)|\right) d s \\
\leq & \left(\omega_{1}+\rho\|w\|_{\infty}+L\|w\|_{\infty} \max _{(x, Q) \in \mathcal{D}}\left|\beta_{Q}(x, Q)\right|\|\hat{u}\|_{\mathcal{L}^{\infty}((0, L) \times(0, T))}\right) \int_{0}^{t}\|v(s)\|_{1} d s \\
& +\int_{0}^{t}|C(s)-\widehat{C}(s)| d s \\
\leq & \rho_{1} \int_{0}^{t}\|v(s)\|_{1} d s+\|C-\widehat{C}\|_{\mathcal{L}^{1}(0, T)},
\end{aligned}
$$

where $\rho_{1}=\omega_{1}+\rho\|w\|_{\infty}+L\|w\|_{\infty} \max _{(x, Q) \in \mathcal{D}}\left|\beta_{Q}(x, Q)\right|\|\hat{u}\|_{\mathcal{L}^{\infty}((0, L) \times(0, T))}$.
By (1.2.25) and the above inequality, we have

$$
\|v(t)\|_{1} \leq e^{c_{1} T}\left(\|v(0)\|_{1}+\|C-\widehat{C}\|_{\mathcal{L}^{1}(0, T)}\right)+e^{c_{1} T} \rho_{1} \int_{0}^{t}\|v(s)\|_{1} d s
$$

Using Gronwall inequality, we find

$$
\|v(t)\|_{1} \leq \exp \left(c_{1} T+e^{c_{1} T} \rho_{1} t\right)\left(\|v(0)\|_{1}+\|C-\widehat{C}\|_{\mathcal{L}^{1}(0, T)}\right)
$$

Letting $\lambda=e^{c_{1} T}$ and $\gamma=\rho_{1} e^{c_{1} T}$, we obtain

$$
\|u(\cdot, t)-\hat{u}(\cdot, t)\|_{1} \leq \lambda e^{\gamma t}\left(\|u(\cdot, 0)-\hat{u}(\cdot, 0)\|_{1}+\|C-\widehat{C}\|_{\mathcal{L}^{1}(0, T)}\right) .
$$

which completes the proof.

Theorem 1.2.11 shows that the finite difference solution converges to the unique bounded variation solution of (1.1.1).

### 1.3 Numerical results

In this section we assume that $L=1$. Our first numerical result shows that the condition $g_{Q} \leq 0$ is crucial for the global existence of solutions. In this example, we choose $u^{0}(x)=3 \exp \left(-10(x-0.01)^{2}\right), \alpha=0.2, w(x)=x$ and the parameters $g, m, \beta$ and $C$ as follows:

$$
\begin{array}{ll}
g(x, Q)=5(1-x) Q \exp (-2 Q), & m(x, Q)=Q /(1+Q) \\
\beta=0.2 x \exp (-0.2 Q), & C(t)=0 .
\end{array}
$$

In this case $g_{Q}=5(1-x) \exp (-2 Q)(1-2 Q)$. Hence, $g_{Q} \leq 0$ for $Q \geq 0.5$ and $g_{Q}>0$ for $Q \in[0,0.5)$. In Figure 1.1 the 3-D dynamics of the solution is presented (where $\Delta x=0.01, \Delta t=0.01$ and $T=1.5)$. In Figure 1.2 the function $U_{\Delta x, \Delta t}(x, 1.5)$ is plotted for several values of $\Delta x$ and $\Delta t$. This figure indicates that a Dirac delta measure is forming at $x \approx 0.7$ and $T=1.5$. Hence, the weak solution only exists locally in time. In fact, in [9] it was formally shown that if $g=Q, m=0$, and $\alpha=0$, then $Q=\int_{x}^{L} u(x, t) d x$ satisfies the famous Burger's equation and hence becomes discontinuous in a finite time $T>0$. Clearly, each discontinuity in $Q$ corresponds to a Dirac delta measure in $u$. Therefore, to extend the solution beyond $T$, measure valued solutions have to be considered. For completeness, we present in Figure 1.3 the 2-D distribution $Q(x, 0)$ and $Q(x, 1.5)$ for the above example. Note that $Q(x, 1.5)$ is discontinuous at $x \approx 0.7$. Furthermore, $\max _{[0,1] \times[0,1.5]} Q(x, t)=0.4979<0.5$.

In our second example, we test the performance of the finite difference scheme in approximating the long-time behavior of solutions to (1.1.1). To this end, we choose $u^{0}(x)=3 \exp \left(-10(x-0.5)^{2}\right), \alpha=0.5, w(x)=1$, and the parameters $g, m, \beta$ and $C$ as


Figure 1.1: The 3-D dynamics of the solution $u(x, t)$.


Figure 1.2: The 2-D distribution of the solution $U_{\Delta x, \Delta t}$ at $T=1.5$ for various values of $\Delta x$ and $\Delta t$.


Figure 1.3: The 2-D distribution of $Q(x, 0)$ and $Q(x, 1.5)$.
follows:

$$
\begin{array}{ll}
g(x, Q)=(1-x)\left(3-x+\frac{1}{2} x^{2}-Q\right), & m(x, Q)=4+2 Q+\frac{1}{2}(1-x)^{2}, \\
\beta=\frac{300}{131}(1+x)(2-Q), & C(t)=0 .
\end{array}
$$

One can easily verify that for this choice of parameters a nontrivial solution to the steady-state problem

$$
\begin{aligned}
& (g(x, Q(x)) u(x))_{x}+m(x, Q(x)) u(x)=0 \\
& g(0, Q(0)) u(0)=\int_{0}^{L} \beta(x, Q(x)) u(x) d x
\end{aligned}
$$

is $u^{*}(x)=1-x$.
Our numerical results presented in Figure 1.4 indicate that $u(x, t)$ converges to $u^{*}(x)$ in $\mathcal{L}^{1}$ norm, and in Figure 1.5 we present $u(x, 20)$ which approximates the function $u^{*}(x)$ (where $\triangle x=0.01, \triangle t=0.025$ and $T=20$ ).


Figure 1.4: The $\mathcal{L}^{1}$ norm of $U_{\Delta x, \Delta t}(x, t)-u^{*}(x), t \in[0,20]$


Figure 1.5: The graph of $U_{\Delta x, \Delta t}(x, 20)$.

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## Chapter 2

## Parameter Estimation in a Coupled System of Nonlinear Size-Structured Populations

In this chapter, a least-squares technique is developed for identifying unknown parameters in a coupled system of nonlinear size-structured populations. Convergence results for the parameter estimation technique are established. Ample numerical simulations and statistical evidence are provided to demonstrate the feasibility of this approach.

### 2.1 Introduction

A typical direct problem for structured populations is to use the knowledge of underlying mechanism at individual level such as growth, mortality and reproduction rates to deduce the behavior at population level. This approach has been extensively studied for many kinds of models which include structured and non-structured populations. In practice, however, our knowledge of the vital rates may be incomplete [31]. In fact, in many animal and plant populations the processes at the individual level are not accessible to direct observation [34]. For example, for nonlinear structured models the dependence of reproduction and mortality rates on the total population is sometimes completely unknown [29]. Even for linear structured models, one may not be able to obtain the exact dependence of the vital rates on the age or size structure [31]. In these cases, one resorts to an inverse problem approach, namely to use knowledge about the behavior
at the population level (e.g, observations of total population numbers) to deduce the underlying mechanisms at the individual level.

In recent years many researchers have focused their attention on developing methodologies for solving inverse problems governed by structured population models (e.g, [1][3], [12]-[17], [19]-[20], [22]-[28], [31]-[35]). In what follows, we briefly review some of the recent work on such inverse problems. For age-structured population models, several approaches have been developed to recover unknown individual vital rates. For example, in $[31,33]$ a fixed point iterative technique was developed to determine the death rate from census data on the age distribution of the population. Therein, conditions on the data are given that lead to a unique solution. In [23] the authors formulated the inverse problem as an operator equation and the least squares method is then used to compute its solution. Due to the ill-posedness of the problem, a regularization technique was considered. In addition, the authors prove that the resulting scheme has a convergence rate of Hölder type. However, no numerical results were reported. A least squares approach was also adopted in [19] for a nonlinear age-structured population model to estimate unknown coefficients from a set of fully discrete observations of the population. Although the convergence of the computed minimizers to a minimizer of the least squares problem was established and numerical results were presented, for many real populations it is generally difficult to obtain discrete observations with respect to age, whereas other quantities such as total population number are easily obtained. In [22] a model describing the evolution in time of size/age structured population was considered. A moving finite element method was used to study the identification problem for such a model. Convergence results for the parameter estimation technique were reported. In [27], by writing a linear age-structured model using the cumulative formulation approach (see e.g., [21]), the authors studied the inverse problem of identifying the birth and death rates from data on the total population size and the cumulative number of births. They also provided conditions on the data that guarantee the uniqueness of the solution to the
inverse problem.
For size-structured population models, the least squares approach has been often used for parameter identification. For example, it was used in $[15,16]$ to estimate the growth rate distribution in a linear size-structured population model. A similar technique was subsequently applied to a semilinear size-structured model in [28] where the mortality rate depends on the total population due to competition. In [2] an inverse problem governed by a phytoplankton aggregation model was studied. Convergence and numerical results for identifying the coagulation kernel were provided. Later, this technique was extended to identify parameters in a size-structured population model in [1, 3] where all the individual vital rates (growth, mortality and reproduction) depend on the total population level. Therein, these parameters are identified from a set of observations corresponding to the total population number. A finite difference method was then used to approximate the infinite dimensional problem. Convergence results for the computed parameter estimates to the true parameter were established. To our knowledge, [3]was the first paper to provide convergence results for parameter estimates when the growth rate is a nonlinear function of the total population (i.e., the size-structured model is represented by a quasilinear first order hyperbolic initial boundary value problem).

In this chapter we extend the discussion in [3] to the following coupled system of quasilinear size-structured populations model:

$$
\begin{align*}
& u_{t}^{I}+\left(g^{I}(x, P(t ; q)) u^{I}\right)_{x}+m^{I}(x, P(t ; q)) u^{I}=0, \quad(x, t) \in(0, L] \times(0, T], \\
& g^{I}(0, P(t ; q)) u^{I}(0, t ; q)=C^{I}(t)+\sum_{J=1}^{N} \int_{0}^{L} \gamma^{I, J} \beta^{J}(x, P(t ; q)) u^{J}(x, t ; q) d x, \quad t \in(0, T], \\
& u^{I}(x, 0 ; q)=u^{I, 0}(x), \quad x \in[0, L] . \tag{2.1.1}
\end{align*}
$$

Here $q=\left(q^{1}, q^{2}, \ldots, q^{N}\right)$ with $q^{I}=\left(g^{I}, m^{I}, \beta^{I}, C^{I}\right), I=1,2, \ldots, N$, the parameters to be identified. The function $u^{I}(x, t ; q), I=1,2, \ldots, N$, is the parameter-dependent size density (number per unit size) of individuals in the $I$ th population having size $x$ at time
$t$, and

$$
\begin{equation*}
P(t ; q)=\sum_{J=1}^{N} \int_{0}^{L} u^{J}(x, t ; q) d x \tag{2.1.2}
\end{equation*}
$$

is the total population at time t . The function $g^{I}$ denotes the growth rate of an individual in the $I$ th population, $m^{I}$ denotes the mortality rate of an individual in the $I$ th population, and $\beta^{I}$ is the reproduction rate of an individual in the $I$ th population. The function $C^{I}$ represents the inflow rate of the $I$ th population of zero-size individuals from an external source (e.g., in a tree population model seeds moved by wind).

The model (2.1.1), which was developed by the authors in [4], is a generalization of several size-structured population models (usually referred to as structured models with rate distributions) which have been investigated in [14, 15, 16, 28]. Motivated by the fact that, in addition to observable characteristics such as age or size of the individuals, non-observable genetic characteristics may often play a crucial role in the development of the individuals, researchers in [14] presented the first such generalization of the classical Sinko-Streifer model. This model, which is a linear version of (2.1.1), has vital individual rates that are independent of the total population and distributed over an an infinite-dimensional admissible parameter space with a probability measure. It was shown through numerical simulations in [14] that there is a crucial difference between the dynamics of distributed rate size-structured population models and the classical SinkoStreifer models. In particular, the classical Sinko-Streifer model cannot have dispersion of the density of the population in age or size except under biologically unreasonable conditions on the growth rate [15]. That is why the classical Sinko-Streifer models are in conflict with field data collected by experimental biologists. These data sets show that a population with unimodal distribution evolves into a bimodal distribution (see [14] and [30]). In [17] the authors used least squares approach to fit these distributed rate models to data obtained in [14]. The resulting good fit indicates that the need for such modification is crucial if these models were to be used as prediction tools.

In addition to extending the theory in [3] to the coupled quasilinear system (2.1.1), a
main novelty of our current research is that we report on extensive numerical simulations. These simulations are then used to obtain statistical results which provide solid evidence on the feasibility of this approach. It is worth pointing out that with the exception of [25] the above-mentioned articles do not report on any statistical studies.

By a weak solution to problem (2.1.1) we mean a bounded and measurable function $u(x, t ; q)=\left(u^{1}(x, t ; q), u^{2}(x, t ; q), \ldots, u^{N}(x, t ; q)\right)$ satisfying

$$
\begin{align*}
& \int_{0}^{L} u^{I}(x, t ; q) \varphi(x, t) d x-\int_{0}^{L} u^{I}(x, 0 ; q) \varphi(x, 0) d x \\
& \quad=\int_{0}^{t} \int_{0}^{L}\left(u^{I} \varphi_{s}+g^{I} u^{I} \varphi_{x}-m^{I} u^{I} \varphi\right) d x d s  \tag{2.1.3}\\
& \quad+\int_{0}^{t} \varphi(0, s)\left(C^{I}(s)+\sum_{J=1}^{N} \int_{0}^{L} \gamma^{I, J} \beta^{J}(x, P(s ; q)) u^{J}(x, s ; q) d x\right) d s
\end{align*}
$$

for $t \in[0, T], I=1,2, \ldots, N$, and every test function $\varphi \in \mathcal{C}^{1}([0, L] \times[0, T])$.
We first impose a condition on the initial data: for any $I=1,2, \ldots, N$
(H1) $u^{I, 0} \in B V[0, L]$ and $u^{I, 0}(x) \geq 0$.
Then let $B=\prod_{I=1}^{N} B^{I}$ with $B^{I}=\mathcal{C}^{1}\left([0, L] ; \mathcal{C}_{b}[0, \infty)\right) \times \mathcal{C}_{b}(\Omega) \times \mathcal{C}_{b}(\Omega) \times \mathcal{C}[0, T]$, where $\Omega=[0, L] \times[0, \infty)$ and $\mathcal{C}_{b}(\Omega)$ denotes the space of uniformly bounded continuous functions on $\Omega$. We assume that our admissible parameter space $Q^{I}$ is a compact subset of $B^{I}$ satisfying (H2)-(H5) below.
(H2) $\beta^{I}(x, P)$ is a nonnegative Lipschitz continuous function in $x$ and $P$ with a Lipschitz constant $L_{1}$. Furthermore, $\beta^{I}(x, P) \leq \omega_{1}$, where $\omega_{1}$ is a positive constant.
(H3) $m^{I}(x, P)$ is a nonnegative Lipschitz continuous function in $x$ and $P$ with a Lipschitz constant $L_{2}$. Furthermore, $m^{I}(x, P) \leq \omega_{2}$, where $\omega_{2}$ is a positive constant.
(H4) $g^{I}(x, P)$ is twice continuously differentiable with respect to $x$ and satisfies $\left|g^{I}(x, P)\right|+$ $\left|g_{x}^{I}(x, P)\right|+\left|g_{x x}^{I}(x, P)\right| \leq \omega_{3}$, where $\omega_{3}$ is a positive constant. Furthermore, $g^{I}(x, P)>$ 0 for $x \in[0, L)$ and $g^{I}(L, P)=0$, and $g^{I}(x, P)$ and $g_{x}^{I}(x, P)$ are Lipschitz continuous in $P$ with a Lipschitz constant $L_{3}$.
(H5) $C^{I}(t)$ is a nonnegative Lipschitz continuous function with a Lipschitz constant $L_{4}$. Let $Q=\prod_{I=1}^{N} Q^{I}$, then $Q$ is a compact subset of $B$.

Depending on the values of the constants $0 \leq \gamma^{I, J} \leq 1$, the model (2.1.1) may have two different interpretations. If $\gamma^{I, I}=1$ and $\gamma^{I, J}=0, I \neq J$, the model represents the dynamics of several populations competing for common resources. On the other hand, if $\gamma^{I, J}>0, I, J=1,2, \ldots, N$, then the model may describe the dynamics of one population consisting of $N$ subpopulations, each with its own characteristics. Hence, $\gamma^{I, J}$ represents the probability that an individual of the $J$ th subpopulation will reproduce an individual of the $I$ th subpopulation. Therefore, two different ways for observing data will be considered. These lead to the following two different least-squares functionals to be minimized: The first one is based on the assumption that the model (2.1.1) describes $N$ different competing populations. Hence observations $Z_{I, k}$ which correspond to the total number of individuals in the $I$ th population at time $t_{k}$ are assumed to be available (this case corresponds to $\gamma^{I, I}=1$ and $\left.\gamma^{I, J}=0, I \neq J\right)$. We define the least-squares cost functional for this case to be

$$
\begin{equation*}
\mathcal{J}(q)=\sum_{I} \sum_{k}\left|\log \left(\int_{0}^{L} u^{I}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{I, k}+1\right)\right|^{2}, \tag{2.1.4}
\end{equation*}
$$

which is minimized over $Q$. The other case assumes that (2.1.1) models one species which has been divided into $N$ not readily distinguishable subpopulations. In this case, we assume that we can only observe aggregate data $Z_{k}$, the total number of individuals at time $t_{k}$ (this case corresponds to $\gamma^{I, J}>0, I, J=1,2, \ldots, N$ ). We define the least-squares cost functional

$$
\begin{equation*}
\mathcal{J}(q)=\sum_{k}\left|\log \left(\sum_{I} \int_{0}^{L} u^{I}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{k}+1\right)\right|^{2} \tag{2.1.5}
\end{equation*}
$$

which is minimized over $Q$.
We remark that minimizing (2.1.4) over $Q$ is equivalent to the maximum likelihood
estimation of $q$ if

$$
\epsilon_{I, k}=\log \left(\int_{0}^{L} u^{I}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{I, k}+1\right)
$$

are i.i.d. normal, and minimizing (2.1.5) over $Q$ is equivalent to the maximum likelihood estimation of $q$ if

$$
\epsilon_{k}=\log \left(\sum_{I} \int_{0}^{L} u^{I}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{k}+1\right)
$$

are i.i.d. normal.
The remainder of this chapter is organized as follows. In Section 2.2, we present a finite difference scheme for computing the solution of (2.1.1) and then provide convergence results for the parameter estimation technique. In Section 2.3, we give ample numerical and statistical results. Some concluding remarks are made in Section 2.4.

### 2.2 Approximation Scheme and Convergence Theory

The following notation will be used throughout this chapter: $\Delta x=L / n$ and $\Delta t=T / l$ denote the spatial and time mesh size, respectively. The mesh points are given by $x_{j}=$ $j \Delta x, j=0,1,2, \ldots, n$ and $t_{k}=k \Delta t, k=0,1,2, \ldots, l$. We denote by $u_{j}^{I, k}(q)$ and $P^{k}(q)$ the finite difference approximation of $u^{I}\left(x_{j}, t_{k} ; q\right)$ and $P\left(t_{k} ; q\right)$, respectively, and we let

$$
\begin{aligned}
& g_{j}^{I, k}=g^{I}\left(x_{j}, P^{k}(q)\right), \beta_{j}^{I, k}=\beta^{I}\left(x_{j}, P^{k}(q)\right), \\
& m_{j}^{I, k}=m^{I}\left(x_{j}, P^{k}(q)\right), \text { and } C^{I, k}=C^{I}\left(t_{k}\right) .
\end{aligned}
$$

We define the difference operator

$$
D_{h}^{-}\left(u_{j}^{I, k}\right)=\frac{u_{j}^{I, k}-u_{j-1}^{I, k}}{\Delta x}, \quad 1 \leq j \leq n
$$

and the $\ell^{1}, \ell^{\infty}$ and the $B V$ norms of $u^{I, k}$ by

$$
\left\|u^{I, k}\right\|_{1}=\sum_{j=1}^{n}\left|u_{j}^{I, k}\right| \triangle x, \quad\left\|u^{I, k}\right\|_{\infty}=\max _{0 \leq j \leq n}\left|u_{j}^{I, k}\right|, \quad\left\|u^{I, k}\right\|_{B V}=\sum_{j=1}^{n}\left|D_{h}^{-}\left(u_{j}^{I, k}\right)\right| \triangle x .
$$

We then discretize the partial differential equation in (2.1.1) using the following implicit finite difference approximation

$$
\begin{align*}
& \frac{u_{j}^{I, k+1}(q)-u_{j}^{I, k}(q)}{\Delta t}+\frac{g_{j}^{I, k} u_{j}^{I, k+1}(q)-g_{j-1}^{I, k} u_{j-1}^{I, k+1}(q)}{\triangle x}+m_{j}^{I, k} u_{j}^{I, k+1}(q)=0,1 \leq j \leq n, \\
& g_{0}^{I, k} u_{0}^{I, k+1}(q)=C^{I, k}+\sum_{J=1}^{N} \sum_{j=1}^{n} \gamma^{I, J} \beta_{j}^{J, k} u_{j}^{J, k}(q) \triangle x  \tag{2.2.1}\\
& P^{k+1}(q)=\sum_{I=1}^{N} \sum_{j=1}^{n} u_{j}^{I, k+1}(q) \Delta x
\end{align*}
$$

with the initial condition

$$
u_{j}^{I, 0}=\frac{1}{\triangle x} \int_{(j-1) \Delta x}^{j \Delta x} u^{I, 0}(x) d x, \quad j=1,2, \ldots, n .
$$

If we define

$$
d_{j}^{I, k}=1+\frac{\Delta t}{\triangle x} g_{j}^{I, k}+\triangle t m_{j}^{I, k} \quad j=1,2, \ldots, n, \quad I=1,2, \ldots, N
$$

then (2.2.1) can be equivalently written as the following system of linear equations for

$$
\begin{gather*}
\vec{u}^{k+1}(q)=\left[u_{0}^{1, k+1}(q), u_{1}^{1, k+1}(q), \ldots, u_{n}^{1, k+1}(q), u_{0}^{2, k+1}(q), u_{1}^{2, k+1}(q), \ldots, u_{n}^{2, k+1}(q), \ldots\right. \\
\left.u_{0}^{N, k+1}(q), u_{1}^{N, k+1}(q), \ldots, u_{n}^{N, k+1}(q)\right]^{T} \in \mathbb{R}^{N \times(n+1)} \\
A^{k} \vec{u}^{k+1}(q)=\vec{f}^{k}(q) \tag{2.2.2}
\end{gather*}
$$

where

$$
\begin{aligned}
\vec{f}^{k}(q)= & {\left[C^{1, k}+\sum_{J=1}^{N} \sum_{j=1}^{n} \gamma^{1, J} \beta_{j}^{J, k} u_{j}^{J, k}(q) \triangle x, u_{1}^{1, k}(q), \ldots, u_{n}^{1, k}(q),\right.} \\
& C^{2, k}+\sum_{J=1}^{N} \sum_{j=1}^{n} \gamma^{2, J} \beta_{j}^{J, k} u_{j}^{J, k}(q) \triangle x, u_{1}^{2, k}(q), \ldots, u_{n}^{2, k}(q), \ldots, \\
& \left.C^{N, k}+\sum_{J=1}^{N} \sum_{j=1}^{n} \gamma^{N, J} \beta_{j}^{J, k} u_{j}^{J, k}(q) \triangle x, u_{1}^{N, k}(q), \ldots, u_{n}^{N, k}(q)\right]^{T}
\end{aligned}
$$

and $A^{k}$ is the following block diagonal matrix:

$$
A^{k}=\left(\begin{array}{ccccc}
A^{1, k} & 0 & 0 & \cdots & 0 \\
0 & A^{2, k} & 0 & \cdots & 0 \\
0 & 0 & A^{3, k} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & A^{N, k}
\end{array}\right)
$$

with the lower triangular matrix

$$
A^{I, k}=\left(\begin{array}{cccccc}
g_{0}^{I, k} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{\Delta t}{\Delta x} g_{0}^{I, k} & d_{1}^{I, k} & 0 & \cdots & 0 & 0 \\
0 & -\frac{\Delta t}{\Delta x} g_{1}^{I, k} & d_{2}^{I, k} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & -\frac{\Delta t}{\Delta x} g_{n-1}^{I, k} & d_{n}^{I, k}
\end{array}\right) .
$$

Note that using the assumptions on our parameters one can easily show that equation (2.2.2) has a unique solution satisfying $\vec{u}^{k+1}(q) \geq 0, k=0,1, \ldots, l-1$.

The above approximation can be extended to a family of functions $\left\{U_{\Delta x, \Delta t}^{I}(x, t ; q)\right\}$
defined by

$$
\begin{gather*}
U_{\Delta x, \Delta t}^{I}(x, t ; q)=u_{j}^{I, k}(q) \quad \text { for } \quad(x, t) \in\left[x_{j-1}, x_{j}\right) \times\left[t_{k-1}, t_{k}\right),  \tag{2.2.3}\\
j=1,2, \ldots, n, \quad k=1,2, \ldots, l, \quad I=1,2, \ldots, N .
\end{gather*}
$$

Since our parameter set is infinite dimensional, a finite dimensional approximation of the parameter space is also necessary for computing minimizers. To this end, we consider the following finite-dimensional approximations of (2.1.4) and (2.1.5), respectively:

$$
\begin{equation*}
\mathcal{J}_{\Delta x, \Delta t}(q)=\sum_{I} \sum_{k}\left|\log \left(\int_{0}^{L} U_{\Delta x, \Delta t}^{I}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{I, k}+1\right)\right|^{2} \tag{2.2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{J}_{\Delta x, \Delta t}(q)=\sum_{k}\left|\log \left(\sum_{I} \int_{0}^{L} U_{\Delta x, \Delta t}^{I}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{k}+1\right)\right|^{2} \tag{2.2.5}
\end{equation*}
$$

each of which is minimized over $Q_{M}$, a compact finite-dimensional approximation of the parameter space $Q$. In order to establish the convergence results for the parameter estimation technique, we use a similar approach to that in [3], which is based on the abstract theory in [18].
Theorem 2.2.1. Let $q^{i}=\left(q^{1, i}, q^{2, i}, \ldots, q^{N, i}\right)$ and suppose that for each $I, q^{I, i} \rightarrow q^{I}$ in $Q^{I}$ and $\Delta x_{i}, \Delta t_{i} \rightarrow 0$ as $i \rightarrow \infty$. Let

$$
U_{\Delta x_{i}, \Delta t_{i}}\left(x, t ; q^{i}\right)=\left(U_{\Delta x_{i}, \Delta t_{i}}^{1}\left(x, t ; q^{i}\right), U_{\Delta x_{i}, \Delta t_{i}}^{2}\left(x, t ; q^{i}\right), \ldots, U_{\Delta x_{i}, \Delta t_{i}}^{N}\left(x, t ; q^{i}\right)\right)
$$

denote the solution of the finite difference scheme, and let

$$
u(x, t ; q)=\left(u^{1}(x, t ; q), u^{2}(x, t ; q), \ldots, u^{N}(x, t ; q)\right)
$$

be the unique weak solution of our problem with initial condition

$$
u^{0}(x)=\left(u^{1,0}(x), u^{2,0}(x), \ldots, u^{N, 0}(x)\right)
$$

and parameter $q$, then $U_{\Delta x_{i}, \Delta t_{i}}^{I}\left(x, t ; q^{i}\right) \rightarrow u^{I}(x, t ; q)$ in $\mathcal{L}^{1}(0, L)$ uniformly in $t \in[0, T]$.

Proof. Define $u_{j}^{I, k, i}=u_{j}^{I, k}\left(q^{i}\right)$. From the fact that $Q^{I}$ is compact and the results of [4], there exist positive constants $c_{1}, c_{2}, c_{3}, c_{4}$ such that for each $I=1,2, \ldots, N$, we have $\sum_{I=1}^{N}\left\|u^{I, k, i}\right\|_{1} \leq c_{1},\left\|u^{I, k, i}\right\|_{\infty} \leq c_{2},\left\|u^{I, k, i}\right\|_{B V} \leq c_{3}$ and $\sum_{j=1}^{n}\left|\frac{u_{j}^{I, r, i}-u_{j}^{I, s, i}}{\Delta t_{i}}\right| \Delta x_{i} \leq c_{4}(r-s)$, where $r>s$. Thus, for each $I$ there exists a $B V([0, L] \times[0, T])$ function $\hat{u}^{I}(x, t)$ such that $U_{\Delta x_{i}, \Delta t_{i}}^{I}\left(x, t ; q^{i}\right) \rightarrow \hat{u}^{I}(x, t)$ in $\mathcal{L}^{1}(0, L)$ uniformly in $t$. Hence, from the uniqueness of bounded variation weak solutions stated in [4], we only need to show that $\hat{u}(x, t)=$ $\left(\hat{u}^{1}(x, t), \hat{u}^{2}(x, t), \ldots, \hat{u}^{N}(x, t)\right)$ is the weak solution corresponding to the parameter $q$. To this end, we multiply the first equation of (2.2.1) by $\varphi_{j}^{k+1}=\varphi\left(x_{j}, t_{k+1}\right)$, where $\varphi \in$ $\mathcal{C}^{1}([0, L] \times[0, T])$, to obtain

$$
\begin{aligned}
& \frac{u_{j}^{I, k+1, i} \varphi_{j}^{k+1}-u_{j}^{I, k, i} \varphi_{j}^{k}}{\Delta t_{i}}-u_{j}^{I, k, i} \frac{\varphi_{j}^{k+1}-\varphi_{j}^{k}}{\Delta t_{i}}+\frac{g_{j}^{I, k, i} u_{j}^{I, k+1, i} \varphi_{j}^{k+1}-g_{j-1}^{I, k, i} u_{j-1}^{I, k+1, i} \varphi_{j-1}^{k+1}}{\Delta x_{i}} \\
& -g_{j-1}^{I, k, i} u_{j-1}^{I, k+1, i} \frac{\varphi_{j}^{k+1}-\varphi_{j-1}^{k+1}}{\Delta x_{i}}+m_{j}^{I, k, i} u_{j}^{I, k+1, i} \varphi_{j}^{k+1}=0 .
\end{aligned}
$$

Multiplying the above equality both sides by $\Delta x_{i} \Delta t_{i}$ and summing over $j=1,2, \ldots, n$, $k=0,1, \ldots, l-1$, we find

$$
\begin{aligned}
& \sum_{j=1}^{n}\left(u_{j}^{I, l, i} \varphi_{j}^{l}-u_{j}^{I, 0, i} \varphi_{j}^{0}\right) \Delta x_{i}-\sum_{k=0}^{l-1} \sum_{j=1}^{n} u_{j}^{I, k, i} \frac{\varphi_{j}^{k+1}-\varphi_{j}^{k}}{\Delta t_{i}} \Delta x_{i} \Delta t_{i} \\
& +\sum_{k=0}^{l-1} \frac{g_{n}^{I, k, i} u_{n}^{I, k+1, i} \varphi_{n}^{k+1}-g_{0}^{I, k, i} u_{0}^{I, k+1, i} \varphi_{0}^{k+1}}{\Delta x_{i}} \Delta x_{i} \Delta t_{i} \\
& -\sum_{k=0}^{l-1} \sum_{j=1}^{n} g_{j-1}^{I, k, i} u_{j-1}^{I, k+1, i} \frac{\varphi_{j}^{k+1}-\varphi_{j-1}^{k+1}}{\Delta x_{i}} \Delta x_{i} \Delta t_{i}+\sum_{k=0}^{l-1} \sum_{j=1}^{n} m_{j}^{I, k, i} u_{j}^{I, k+1, i} \varphi_{j}^{k+1} \Delta x_{i} \Delta t_{i}=0 .
\end{aligned}
$$

Since $g_{n}^{I, k, i}=0$ and $q^{I, i} \rightarrow q^{I}$ as $i \rightarrow \infty$ in $Q^{I}$, passing to the limit we have

$$
\begin{aligned}
& \int_{0}^{L} \hat{u}^{I}(x, t) \varphi(x, t) d x-\int_{0}^{L} \hat{u}^{I}(x, 0) \varphi(x, 0) d x \\
= & \int_{0}^{t} \int_{0}^{L}\left(\hat{u}^{I} \varphi_{s}+g^{I} \hat{u}^{I} \varphi_{x}-m^{I} \hat{u}^{I} \varphi\right) d x d s \\
& +\int_{0}^{t} \varphi(0, s)\left(C^{I}(s)+\sum_{J=1}^{N} \int_{0}^{L} \gamma^{I, J} \beta^{J}(x, P(s)) \hat{u}^{J}(x, s) d x\right) d s .
\end{aligned}
$$

Thus, $\hat{u}(x, t)$ is the weak solution corresponding to the parameter $q$.

Since the logarithm function is continuous on $[1, \infty)$, as an immediate consequence of Theorem 2.2.1, we obtain the following:

Corollary 2.2.2. Let $U_{\Delta x, \Delta t}$ denote the numerical solution of (2.2.1) with parameter $q^{i} \rightarrow q$ and $\Delta x_{i}, \Delta t_{i} \rightarrow 0$. Then

$$
\mathcal{J}_{\Delta x_{i}, \Delta t_{i}}\left(q^{i}\right) \rightarrow \mathcal{J}(q), \quad \text { as } i \rightarrow \infty .
$$

In the next theorem, we establish the continuity of the approximate cost functional, so that the computational problem of finding approximate minimizer is well-posed.

Theorem 2.2.3. Let $\Delta x$ and $\Delta t$ be fixed. For each $q^{I} \in Q^{I}$, let $U_{\Delta x, \Delta t}^{I}(x, t ; q)$ denote the solution of the finite difference scheme, and $q^{I, i} \rightarrow q^{I}$ as $i \rightarrow \infty$ in $Q^{I}$, then $U_{\Delta x, \Delta t}^{I}\left(x, t ; q^{i}\right) \rightarrow U_{\Delta x, \Delta t}^{I}(x, t ; q)$ as $i \rightarrow \infty$ in $\mathcal{L}^{1}(0, L)$ uniformly in $t \in[0, T]$.

Proof. Define $\left\{u_{j}^{I, k, i}\right\}$ and $\left\{u_{j}^{I, k}\right\}$ to be the solution of the finite difference scheme with parameter $q^{i}$ and $q$, respectively. Let $v_{j}^{I, k, i}=u_{j}^{I, k, i}-u_{j}^{I, k}$, then $v_{j}^{I, k, i}$ satisfies the following:

$$
\begin{align*}
& \frac{v_{j}^{I, k+1, i}-v_{j}^{I, k, i}}{\Delta t}+D_{h}^{-}\left[g^{I, i}\left(x_{j}, P^{k, i}\right) u_{j}^{I, k+1, i}-g^{I}\left(x_{j}, P^{k}\right) u_{j}^{I, k+1}\right]  \tag{2.2.6}\\
& +m^{I, i}\left(x_{j}, P^{k, i}\right) v_{j}^{I, k+1, i}+\left[m^{I, i}\left(x_{j}, P^{k, i}\right)-m^{I}\left(x_{j}, P^{k}\right)\right] u_{j}^{I, k+1}=0
\end{align*}
$$

for $1 \leq j \leq n$, and

$$
\begin{align*}
& g^{I, i}\left(0, P^{k, i}\right) u_{0}^{I, k+1, i}-g^{I}\left(0, P^{k}\right) u_{0}^{I, k+1} \\
= & C^{I, i}\left(t_{k}\right)-C^{I}\left(t_{k}\right)+\sum_{J=1}^{N} \sum_{j=1}^{n} \gamma^{I, J} \beta^{J, i}\left(x_{j}, P^{k, i}\right) v_{j}^{J, k, i} \Delta x  \tag{2.2.7}\\
& +\sum_{J=1}^{N} \sum_{j=1}^{n} \gamma^{I, J}\left[\beta^{J, i}\left(x_{j}, P^{k, i}\right)-\beta^{J}\left(x_{j}, P^{k}\right)\right] u_{j}^{J, k} \Delta x
\end{align*}
$$

where $P^{k, i}$ denotes $P^{k}\left(q^{i}\right)$. Multiplying both sides of (2.2.6) by $\operatorname{sgn}\left(v_{j}^{I, k+1, i}\right) \Delta x$ and summing over $j=1,2, \ldots, n$, we obtain

$$
\begin{align*}
& \frac{\left\|v^{I, k+1, i}\right\|_{1}-\left\|v^{I, k, i}\right\|_{1}}{\Delta t} \\
\leq & -\sum_{j=1}^{n} D_{h}^{-}\left[g^{I, i}\left(x_{j}, P^{k, i}\right) u_{j}^{I, k+1, i}-g^{I}\left(x_{j}, P^{k}\right) u_{j}^{I, k+1}\right] \operatorname{sgn}\left(v_{j}^{I, k+1, i}\right) \Delta x \\
& -\sum_{j=1}^{n} m^{I, i}\left(x_{j}, P^{k, i}\right)\left|v_{j}^{I, k+1, i}\right| \Delta x  \tag{2.2.8}\\
& -\sum_{j=1}^{n}\left[m^{I, i}\left(x_{j}, P^{k, i}\right)-m^{I}\left(x_{j}, P^{k}\right)\right] u_{j}^{I, k+1} \operatorname{sgn}\left(v_{j}^{I, k+1, i}\right) \Delta x .
\end{align*}
$$

Using the fact for any $a_{j}$ with $a_{j} \geq 0, j=0,1,2, \ldots, n$, we have

$$
\sum_{j=1}^{n} D_{h}^{-}\left(a_{j} b_{j}\right) \operatorname{sgn}\left(b_{j}\right) \Delta x \geq a_{n}\left|b_{n}\right|-a_{0}\left|b_{0}\right|
$$

we obtain

$$
\begin{align*}
& -\sum_{j=1}^{n} D_{h}^{-}\left[g^{I, i}\left(x_{j}, P^{k, i}\right) u_{j}^{I, k+1, i}-g^{I}\left(x_{j}, P^{k}\right) u_{j}^{I, k+1}\right] \operatorname{sgn}\left(v_{j}^{I, k+1, i}\right) \Delta x \\
= & -\sum_{j=1}^{n} D_{h}^{-}\left(g^{I, i}\left(x_{j}, P^{k, i}\right) v_{j}^{I, k+1, i}\right) \operatorname{sgn}\left(v_{j}^{I, k+1, i}\right) \Delta x \\
& -\sum_{j=1}^{n} D_{h}^{-}\left[\left(\left(g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I}\left(x_{j}, P^{k}\right)\right) u_{j}^{I, k+1}\right] \operatorname{sgn}\left(v_{j}^{I, k+1, i}\right) \Delta x\right.  \tag{2.2.9}\\
\leq & g^{I, i}\left(0, P^{k, i}\right)\left|v_{0}^{I, k+1, i}\right|+\sup _{1 \leq j \leq n}\left|g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I}\left(x_{j}, P^{k}\right)\right|\left\|u^{I, k+1}\right\|_{B V} \\
& +\sup _{1 \leq j \leq n}\left|D_{h}^{-}\left(g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I}\left(x_{j}, P^{k}\right)\right)\right|\left(\left\|u^{I, k+1}\right\|_{\infty}+\left(\left\|u^{I, k+1}\right\|_{1}\right) .\right.
\end{align*}
$$

By (2.2.7), we have

$$
\begin{align*}
& g^{I, i}\left(0, P^{k, i}\right)\left|v_{0}^{I, k+1, i}\right| \\
\leq & \left|g^{I, i}\left(0, P^{k, i}\right)-g^{I}\left(0, P^{k}\right)\right| u_{0}^{I, k+1}+\left|C^{I, i}\left(t_{k}\right)-C^{I}\left(t_{k}\right)\right| \\
& +\omega_{1} \sum_{J=1}^{N}\left\|v^{J, k, i}\right\|_{1}+\max _{1 \leq J \leq N} \sup _{1 \leq j \leq n}\left|\beta^{J, i}\left(x_{j}, P^{k, i}\right)-\beta^{J}\left(x_{j}, P^{k}\right)\right| \sum_{J=1}^{N}\left\|u^{J, k}\right\|_{1} . \tag{2.2.10}
\end{align*}
$$

Summing (2.2.8) over $I=1,2, \ldots, N$, and using (2.2.9) and (2.2.10), we obtain

$$
\begin{aligned}
& \frac{\sum_{I=1}^{N}\left\|v^{I, k+1, i}\right\|_{1}-\sum_{I=1}^{N}\left\|v^{I, k, i}\right\|_{1}}{\Delta t} \\
\leq & \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|D_{h}^{-}\left(g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I}\left(x_{j}, P^{k}\right)\right)\right|\left(N \max _{1 \leq I \leq N}\left\|u^{I, k+1}\right\|_{\infty}+\sum_{I=1}^{N}\left\|u^{I, k+1}\right\|_{1}\right) \\
& +N \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I}\left(x_{j}, P^{k}\right)\right| \max _{1 \leq I \leq N}\left\|u^{I, k+1}\right\|_{B V} \\
& +N \max _{1 \leq I \leq N}\left|g^{I, i}\left(0, P^{k, i}\right)-g^{I}\left(0, P^{k}\right)\right| \max _{1 \leq I \leq N}\left\|u^{I, k+1}\right\|_{\infty}+N \max _{1 \leq I \leq N}\left|C^{I, i}\left(t_{k}\right)-C^{I}\left(t_{k}\right)\right| \\
& +N \max _{1 \leq J \leq N} \sup _{1 \leq j \leq n}\left|\beta^{J, i}\left(x_{j}, P^{k, i}\right)-\beta^{J}\left(x_{j}, P^{k}\right)\right| \sum_{J=1}^{N}\left\|u^{J, k}\right\|_{1}+N \omega_{1} \sum_{I=1}^{N}\left\|v^{I, k, i}\right\|_{1} \\
& +\max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|m^{I, i}\left(x_{j}, P^{k, i}\right)-m^{I}\left(x_{j}, P^{k}\right)\right| \sum_{I=1}^{N}\left\|u^{I, k+1}\right\|_{1} .
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
& \left|g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I}\left(x_{j}, P^{k}\right)\right| \\
\leq & \left|g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I, i}\left(x_{j}, P^{k}\right)\right|+\left|g^{I, i}\left(x_{j}, P^{k}\right)-g^{I}\left(x_{j}, P^{k}\right)\right|
\end{aligned}
$$

we have from (H4) the following:

$$
\begin{aligned}
& \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I}\left(x_{j}, P^{k}\right)\right| \\
\leq & L_{3} \sum_{I=1}^{N}\left\|v^{I, k, i}\right\|_{1}+\max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|g^{I, i}\left(x_{j}, P^{k}\right)-g^{I}\left(x_{j}, P^{k}\right)\right| .
\end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
& \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|\beta^{I, i}\left(x_{j}, P^{k, i}\right)-\beta^{I}\left(x_{j}, P^{k}\right)\right| \\
\leq & L_{1} \sum_{I=1}^{N}\left\|v^{I, k, i}\right\|_{1}+\max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|\beta^{I, i}\left(x_{j}, P^{k}\right)-\beta^{I}\left(x_{j}, P^{k}\right)\right|
\end{aligned}
$$

and

$$
\begin{aligned}
& \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|m^{I, i}\left(x_{j}, P^{k, i}\right)-m^{I}\left(x_{j}, P^{k}\right)\right| \\
\leq & L_{2} \sum_{I=1}^{N}\left\|v^{I, k, i}\right\|_{1}+\max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|m^{I, i}\left(x_{j}, P^{k}\right)-m^{I}\left(x_{j}, P^{k}\right)\right| .
\end{aligned}
$$

Furthermore, straightforward computations yield

$$
\begin{aligned}
& \left|D_{h}^{-}\left[g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I}\left(x_{j}, P^{k}\right)\right]\right| \\
= & \left|\frac{1}{\Delta x}\left(\int_{0}^{1} \frac{d}{d r} g^{I, i}\left(r x_{j}+(1-r) x_{j-1}, P^{k, i}\right) d r-\int_{0}^{1} \frac{d}{d r} g^{I}\left(r x_{j}+(1-r) x_{j-1}, P^{k}\right) d r\right)\right| \\
= & \left|\int_{0}^{1} g_{x}^{I, i}\left(r x_{j}+(1-r) x_{j-1}, P^{k, i}\right) d r-\int_{0}^{1} g_{x}^{I}\left(r x_{j}+(1-r) x_{j-1}, P^{k}\right) d r\right| \\
\leq & \int_{0}^{1}\left|g_{x}^{I, i}\left(r x_{j}+(1-r) x_{j-1}, P^{k, i}\right)-g_{x}^{I, i}\left(r x_{j}+(1-r) x_{j-1}, P^{k}\right)\right| d r \\
& +\int_{0}^{1}\left|g_{x}^{I, i}\left(r x_{j}+(1-r) x_{j-1}, P^{k}\right)-g_{x}^{I}\left(r x_{j}+(1-r) x_{j-1}, P^{k}\right)\right| d r .
\end{aligned}
$$

Hence, from (H4) we obtain

$$
\begin{aligned}
& \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|D_{h}^{-}\left[g^{I, i}\left(x_{j}, P^{k, i}\right)-g^{I}\left(x_{j}, P^{k}\right)\right]\right| \\
\leq & L_{3} \sum_{I=1}^{N}\left\|v^{I, k, i}\right\|_{1}+\max _{1 \leq I \leq N} \sup _{1 \leq j \leq n} \int_{0}^{1}\left|g_{x}^{I, i}\left(\bar{x}_{j}, P^{k}\right)-g_{x}^{I}\left(\bar{x}_{j}, P^{k}\right)\right| d r
\end{aligned}
$$

where $\bar{x}_{j}=r x_{j}+(1-r) x_{j-1}$. Set

$$
\begin{aligned}
\delta_{k}= & L_{3}\left(N \max _{1 \leq I \leq N}\left\|u^{I, k+1}\right\|_{\infty}+\sum_{I=1}^{N}\left\|u^{I, k+1}\right\|_{1}\right)+N L_{1} \sum_{I=1}^{N}\left\|u^{I, k}\right\|_{1}+N \omega_{1} \\
& +N L_{3}\left(\max _{1 \leq I \leq N}\left\|u^{I, k+1}\right\|_{B V}+\max _{1 \leq I \leq N}\left\|u^{I, k+1}\right\|_{\infty}\right)+L_{2} \sum_{I=1}^{N}\left\|u^{I, k+1}\right\|_{1}
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_{k, i}= & \left(N \max _{1 \leq I \leq N}\left\|u^{I, k+1}\right\|_{\infty}+\sum_{I=1}^{N}\left\|u^{I, k+1}\right\|_{1}\right) \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n} \int_{0}^{1}\left|g_{x}^{I, i}\left(\bar{x}_{j}, P^{k}\right)-g_{x}^{I}\left(\bar{x}_{j}, P^{k}\right)\right| d r \\
& +N \max _{1 \leq I \leq N}\left\|u^{I, k+1}\right\|_{B V} \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|g^{I, i}\left(x_{j}, P^{k}\right)-g^{I}\left(x_{j}, P^{k}\right)\right| \\
& +N \max _{1 \leq I \leq N}\left\|u^{I, k+1}\right\|_{\infty} \max _{1 \leq I \leq N}\left|g^{I, i}\left(0, P^{k}\right)-g^{I}\left(0, P^{k}\right)\right|+N \max _{1 \leq I \leq N}\left|C^{I, i}\left(t_{k}\right)-C^{I}\left(t_{k}\right)\right| \\
& +N \sum_{I=1}^{N}\left\|u^{I, k}\right\|_{1} \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|\beta^{I, i}\left(x_{j}, P^{k}\right)-\beta^{I}\left(x_{j}, P^{k}\right)\right| \\
& +\sum_{I=1}^{N}\left\|u^{I, k+1}\right\|_{1} \max _{1 \leq I \leq N} \sup _{1 \leq j \leq n}\left|m^{I, i}\left(x_{j}, P^{k}\right)-m^{I}\left(x_{j}, P^{k}\right)\right| .
\end{aligned}
$$

Then, we have

$$
\frac{\sum_{I=1}^{N}\left\|v^{I, k+1, i}\right\|_{1}-\sum_{I=1}^{N}\left\|v^{I, k, i}\right\|_{1}}{\Delta t} \leq \delta_{k} \sum_{I=1}^{N}\left\|v^{I, k, i}\right\|_{1}+\rho_{k, i}
$$

Since for each $k, \rho_{k, i} \rightarrow 0$ as $i \rightarrow \infty$, the desired result easily follows from this inequality.

Theorem 2.2.4. Suppose that $Q_{M}$ is a sequence of compact subsets of $Q$. Moreover, assume that for each $q \in Q$, there exists a sequence of $q_{M} \in Q_{M}$ such that $q_{M} \rightarrow q$ as $M \rightarrow \infty$. Then the functional $\mathcal{J}_{\Delta x, \Delta t}$ has a minimizer over $Q_{M}$. Furthermore, if $q_{M}^{i}$ denotes a minimizer of $\mathcal{J}_{\Delta x_{i}, \Delta t_{i}}$ over $Q_{M}$ and $\Delta x_{i}, \Delta t_{i} \rightarrow 0$, then any subsequence of $q_{M}^{i}$ has a further subsequence which converges to a minimizer of $\mathcal{J}$.

Proof. The proof of this theorem is a direct application of the abstract theory in [18],
based on the convergence of $\mathcal{J}_{\Delta x_{i}, \Delta t_{i}}\left(q^{i}\right) \rightarrow \mathcal{J}(q)$.

### 2.3 Numerical Results

In this section, we present ample numerical simulations and statistical results. In all of the simulations below we assume $L=1, T=1$, and $C^{I}(t)=0$ for $I=1,2, \ldots, N$.

In subsections 3.1 and 3.2, we assume $N=1$ and that all the parameters are known except for $\beta$. To estimate $\beta$ we use data which are generated computationally as follows: Let

$$
\begin{aligned}
& u^{0}(x)=3 \exp \left(-2(x-0.5)^{2}\right), \quad g(x, P)=5(1-x) \exp (-3 P) \\
& m(x, P)=\exp \left(4(x-0.4)^{2}\right) \exp (0.2 P), \quad \beta(x, P)=6 x(1-x) \exp (-3 P)
\end{aligned}
$$

and we solve (2.2.1) and (2.2.3) for $U_{\Delta x, \Delta t}(x, t)$. We set the data $Z_{k}=\left(1+\varepsilon_{k}\right) \int_{0}^{1} U_{\Delta x, \Delta t}\left(x, t_{k}\right) d x$, where $\varepsilon_{k}$ is a random sample from a normal random number generator with mean zero and standard deviation $\sigma=0.02$.

### 2.3.1 $1-D$ linear estimation problem for finite dimensional parameter space when $N=1$

In our first example we assume that $\beta$ is of a separable form given by $\beta(x, P)=$ $b(x) \exp (-3 P)$, where $b(x)=\mu x\left(1-x^{\nu}\right)$ with $\mu$ and $\nu$ two unknown constants to be identified. Hence, the solution to our least-squares problems involves identifying the two constants $\mu$ and $\nu$ from a compact subset of $\mathbb{R}_{+}^{2}$ so as to minimize the least-squares cost functional

$$
\mathcal{J}_{\Delta x, \Delta t}(q)=\sum_{k=1}^{m}\left|\log \left(\int_{0}^{1} U_{\Delta x, \Delta t}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{k}+1\right)\right|^{2}
$$

In order to test the performance of the parameter-estimation technique when no infinite dimensional effects are present, in Figure 2.1 we choose $\Delta x=\Delta t=0.005$ for both generating the data and the numerical solution (2.2.3) in the least-squares problem. This avoids the infinite-dimensional effect of the partial differential equation given in (2.1.1). In fact, if the noise is removed from the data, and the parameters $\mu$ and $\nu$ are known, then numerically solving our model produces the exact data.

In Figure 2.2 we use $\Delta x=\Delta t=0.005$ to generate the data while we use $\Delta x=$ $\Delta t=0.01$ for the numerical solution (2.2.3) in the least-squares problem. Thus, in this case the data are not exactly attained by our model even if the noise is removed (an error is present due to the finite-dimensional approximation of our infinite-dimensional model). The results of Figure 2.2 are obtained by using the same values for the rest of the parameters as those of Figure 2.1.

A similar format for presenting the results of 1000 inverse problem calculations was used in Figure 2.1 and 2.2. The left part of each of the figures represents the $S$ (for our case $S=1000$ ) numerical results for the estimated parameter $b^{s}(x)(s=1,2, \ldots, S)$ versus the exact $b(x)$, where these 1000 distinct numerical results graphed were obtained by solving 1000 inverse problems, each of which corresponds to a given noise sample $\left\{\epsilon_{k}\right\}$. The right part represents the figure of the corresponding $95 \%$ confidence interval (dashed line) versus the exact $b(x)$ (solid line), where the $95 \%$ confidence interval is obtained by choosing the band between the upper $2.5 \%$ and lower $2.5 \%$ of these 1000 numerical results. Table 1 provides statistical results for the corresponding graphs, where $A B(x)=$ $\frac{1}{S} \sum_{s=1}^{S}\left(b^{s}(x)-b(x)\right)$ denotes the average bias for all approximations at $x, R A B(x)=$ $100 \frac{A B(x)}{b(x)}$ denotes the relative average bias for all approximations at $x$ and $S E(x)=$ $\left[\frac{1}{S-1} \sum_{s=1}^{S}\left(b^{s}(x)-b(x)-A B(x)\right)^{2}\right]^{\frac{1}{2}}$ denotes the standard error at the point $x$.

Although the estimates in both figures are good, the results in Figures 2.1-2.2 and Table 2.1 suggest that infinite-dimensional effects can lead to a slightly under biased estimator. We suspect that this bias depends on the choice of the numerical scheme
used for solving the infinite-dimensional partial differential equation model. Here we are using an upwind scheme for approximating the model and a right-hand sum for approximating all the integrals involved. This biased estimator may be improved if, for example, a centered finite difference approximation is used together with a trapezoidal rule for integration.



Figure 2.1: $\Delta x=\Delta t=0.005$ to generate the data and solve the least-squares. For the left part of the figure, each of the grey lines (....) denotes a distinct result for a given sample $\left\{\epsilon_{k}\right\}$.


Figure 2.2: $\Delta x=\Delta t=0.005$ to generate the data and $\Delta x=\Delta t=0.01$ to solve the least-squares. For the left part of the figure, each of the grey lines (....) denotes a distinct result for a given sample $\left\{\epsilon_{k}\right\}$.

The above statistical results (essentially on how measurement error affects estimates) are based on a large number of numerical simulations (somewhat in the spirit of Bayesian

| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :---: | :---: | :---: |
| 0.1 | -0.0037 | -0.6870 | 0.0749 |
| 0.2 | -0.0092 | -0.9580 | 0.0993 |
| 0.3 | -0.0107 | -0.8463 | 0.0975 |
| 0.4 | -0.0079 | -0.5497 | 0.0860 |
| 0.5 | -0.0021 | -0.1427 | 0.0798 |
| 0.6 | 0.0049 | 0.3378 | 0.0852 |
| 0.7 | 0.0110 | 0.8707 | 0.0926 |
| 0.8 | 0.0138 | 1.4425 | 0.0882 |
| 0.9 | 0.0110 | 2.0444 | 0.0605 |


| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :---: | :---: | :---: |
| 0.1 | -0.0390 | -7.2314 | 0.0747 |
| 0.2 | -0.0651 | -6.7812 | 0.1053 |
| 0.3 | -0.0768 | -6.0949 | 0.1130 |
| 0.4 | -0.0763 | -5.2995 | 0.1124 |
| 0.5 | -0.0666 | -4.4422 | 0.1138 |
| 0.6 | -0.0511 | -3.5460 | 0.1188 |
| 0.7 | -0.0331 | -2.6236 | 0.1202 |
| 0.8 | -0.0162 | -1.6830 | 0.1075 |
| 0.9 | -0.0039 | -0.7294 | 0.0706 |

Table 2.1: Left and right tables are statistical results for Figure 2.1 and Figure 2.2, respectively.
based MCMC calculations used to estimate means and variances in a probabilty distribution from "experimental" data). Any estimate of model parameters from data can also be accompanied by an estimate of uncertainty using standard regression formulations from statistics [36]. Thus, in the remaining part of this subsection, we present a statistical based method to actually compute the variance in the estimated model parameters $q=(\mu, \nu)$.

To perform this analysis, we need to compute the sensitivity matrix

$$
X(q)=\left[\begin{array}{cc}
\frac{P_{\mu}\left(t_{1} ; q\right)}{1+P\left(t_{1} ; q\right)} & \frac{P_{\nu}\left(t_{1} ; q\right)}{1+P\left(t_{1} ; q\right)}  \tag{2.3.1}\\
\frac{P_{\mu}\left(t_{2} ; q\right)}{1+P\left(t_{2} ; q\right)} & \frac{P_{\nu}\left(t_{2} ; q\right)}{1+P\left(t_{2} ; q\right)} \\
\cdots & \ldots \\
\frac{P_{\mu}\left(t_{m} ; q\right)}{1+P\left(t_{m} ; q\right)} & \frac{P_{\nu}\left(t_{m} ; q\right)}{1+P\left(t_{m} ; q\right)}
\end{array}\right] .
$$

Note that we cannot compute $P(t ; q), P_{\mu}(t ; q)$ and $P_{\nu}(t ; q)$ directly from our model. Therefore, we use the difference scheme (2.2.1) to obtain the following approximation of $P(t ; q)$ :

$$
\widehat{P}(t ; q)=\int_{0}^{1} U_{\Delta x, \Delta t}(x, t ; q) d x
$$

Then we use a forward difference approximation for the derivative $P_{\mu}(t ; q)$ and $P_{\nu}(t ; q)$
given by

$$
\widehat{P}_{\mu}(t ; \mu, \nu)=\frac{1}{\Delta \mu}(\widehat{P}(t ; \mu+\Delta \mu, \nu)-\widehat{P}(t ; \mu, \nu))
$$

and

$$
\widehat{P}_{\nu}(t ; q)=\frac{1}{\Delta \nu}(\widehat{P}(t ; \mu, \nu+\Delta \nu)-\widehat{P}(t ; \mu, \nu)) .
$$

Substituting $\widehat{P}\left(t_{i} ; q\right), \widehat{P}_{\mu}\left(t_{i} ; q\right)$ and $\widehat{P}_{\nu}\left(t_{i}, q\right)$ for $P\left(t_{i} ; q\right), P_{\mu}\left(t_{i} ; q\right)$ and $P_{\nu}\left(t_{i} ; q\right)$ in (2.3.1), respectively, we obtain the following approximation of $X(q)$ :

$$
\widehat{X}(q)=\left[\begin{array}{cc}
\frac{\widehat{P}_{\mu}\left(t_{1} ; q\right)}{1+\hat{P}\left(t_{1} ; q\right)} & \frac{\widehat{P}_{\nu}\left(t_{1} ; q\right)}{1+\widehat{P}\left(t_{1} ; q\right)} \\
\frac{\widehat{P}}{\mu}\left(t_{2} ; q\right) & \frac{\widehat{P}_{\nu}\left(t_{2} ; q\right)}{1+\widehat{P}\left(t_{2} ; q\right)} \\
\cdots & \cdots \\
\cdots+\widehat{P}\left(t_{2} ; q\right) \\
\frac{\widehat{P}_{\mu}\left(t_{m} ; q\right)}{1+\widehat{P}\left(t_{m} ; q\right)} & \frac{\widehat{P}_{\nu}\left(t_{m} ; q\right)}{1+\widehat{P}\left(t_{m} ; q\right)}
\end{array}\right] .
$$

Under standard assumptions of classical nonlinear regression theory, we know that if $\hat{\epsilon}_{i} \sim \mathcal{N}\left(0, \sigma^{2}\right)$, where $\hat{\epsilon}_{i}$ is the difference between observation and model at time $t_{i}$, then the least-squares estimate $q^{*}$ is expected to be asymptotically normally distributed. In particular, for large samples, we may assume

$$
\begin{equation*}
q^{*} \sim \mathcal{N}\left[q_{0}, \sigma^{2}\left\{X^{T}\left(q_{0}\right) X\left(q_{0}\right)\right\}^{-1}\right] \tag{2.3.2}
\end{equation*}
$$

where $q_{0}$ is the true vector of parameters and $\sigma^{2}\left\{X^{T}\left(q_{0}\right) X\left(q_{0}\right)\right\}^{-1}$ is the true covariance matrix (see [36], Chapter 2).

Since $q_{0}$ and $\sigma^{2}$ are not available, we follow a standard statistical practice [5]: substitute the computed estimate $q^{*}$ for $q_{0}$ and approximate $\sigma^{2}$ by

$$
\begin{equation*}
\hat{\sigma}^{2}=\frac{1}{m-2} \sum_{j=1}^{m}\left(\log \left(\widehat{P}\left(t_{j} ; q^{*}\right)+1\right)-\log \left(Z_{j}+1\right)\right)^{2} \tag{2.3.3}
\end{equation*}
$$

in (2.3.2) to obtain the standard deviation for our estimates. In particular, if

$$
V=\hat{\sigma}^{2}\left\{\widehat{X}^{T}\left(q^{*}\right) \widehat{X}\left(q^{*}\right)\right\}^{-1}=\left[\begin{array}{ll}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{array}\right]
$$

then we take $\sqrt{V_{11}}$ and $\sqrt{V_{22}}$ to be the standard deviation for parameters $\mu$ and $\nu$, respectively. The following two tables are the standard deviation of $\mu$ and $\nu$ for the results of the first eight numerical simulations of Figure 2.1 and Figure 2.2, respectively.

| $\mu$ | 1.1613 | 1.0494 | 1.0451 | 1.1109 | 1.0864 | 1.4684 | 1.1605 | 1.0512 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\nu$ | 1.2124 | 0.3073 | 0.2999 | 0.2741 | 0.2701 | 1.5555 | 0.2482 | 0.2390 |

Table 2.2: Standard deviation for the results of the first 8 numerical simulations of Figure 2.1.

| $\mu$ | 1.7066 | 1.5636 | 1.6192 | 1.7974 | 1.6389 | 2.8009 | 1.8619 | 1.3893 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\nu$ | 0.7716 | 0.3238 | 0.4838 | 0.1812 | 0.3426 | 2.8685 | 0.3828 | 0.4136 |

Table 2.3: Standard deviation for the results of the first 8 numerical simulations of Figure 2.2.

Table 2.4 provide the average standard deviation of $\mu$ and $\nu$ for the results of all the 1000 numerical simulations of Figure 2.1 and Figure 2.2, respectively. We note that in most practical situations using experimental data, one does not expect to have 1000 experiments performed. But the above procedures will produce estimates of variances even in the case when one has only one data set!

|  | Figure 1 | Figure 2 |
| :---: | :---: | :---: |
| $\mu$ | 1.1921 | 1.9197 |
| $\nu$ | 0.4566 | 0.8572 |

Table 2.4: Average of standard deviation for all the results of the numerical simulations of Figures 2.1-2.2.

### 2.3.2 $1-D$ linear estimation problem for infinite dimensional parameter space when $N=1$

In this example, we assume that $\beta$ is of a separable form given by $\beta(x, P)=b(x) \exp (-3 P)$, where $b(x)$ is an unknown parameter that we want to identify.

Let

$$
\mathcal{D}=\{f \in \mathcal{C}[0,1]:|f(x)-f(y)| \leq K|x-y|, f(0)=f(1)=0\}
$$

Choose the parameter space $Q=\mathcal{D}$. Clearly, by Arzela-Ascoli Theorem [37] $Q$ is compact in $\mathcal{C}[0,1]$. We approximate the infinite dimensional parameter space as follows: For $M$ a positive integer and $b \in Q$, we set

$$
\left(\mathcal{I}_{M} b\right)(x)=\sum_{i=1}^{M-1} b\left(\frac{i}{M}\right) \phi_{M}^{i}(x ; 0,1),
$$

where $\phi_{M}^{i}(x ; 0,1)$ are the linear spline functions on a uniform mesh of the interval $[0,1]$. These are defined by

$$
\phi_{M}^{i}(x ; 0,1)= \begin{cases}1-i+\frac{x}{h}, & (i-1) h \leq x \leq i h \\ 1+i-\frac{x}{h}, & \text { ih } \leq x \leq(i+1) h, \\ 0, & |x-i h| \geq h\end{cases}
$$

where $h=\frac{1}{M}$. It can be readily argued that $\lim _{M \rightarrow \infty} \mathcal{I}_{M} b=b$ in $\mathcal{C}[0,1]$, uniformly in $b$ [8]. Hence, if $b_{M} \in Q_{M}=\mathcal{I}_{M}(Q)$ is given by

$$
b_{M}(x)=\sum_{i=1}^{M-1} \lambda_{M}^{i} \phi_{M}^{i}(x ; 0,1),
$$

then the solution of our finite dimensional identification problem involves identifying the $M-1$ coefficients $\left\{\lambda_{M}^{i}\right\}_{i=1}^{M-1}$ from a compact subset of $\mathbb{R}_{+}^{M-1}$ so as to minimize the least-squares cost functional (2.2.4).

In order to indirectly implement the compactness constraints of $Q$, we use a regular-
ized least squares cost functional of the form

$$
\mathcal{J}_{\Delta x, \Delta t}(q)=\sum_{k=1}^{m}\left|\log \left(\int_{0}^{1} U_{\Delta x, \Delta t}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{k}+1\right)\right|^{2}+\alpha \int_{0}^{1}\left|\frac{d}{d x} b_{M}(x)\right|^{2} d x
$$

where $\alpha>0$ is the regularization parameter.
The left part of each of the following figures again represents the $S(=1000)$ numerical results of the estimated parameter versus the exact parameter $b(x)$. The right part represents the figure of the corresponding $95 \%$ confidence interval (dashed line) versus the exact $b(x)$ (solid line). The tables provide statistical results for the corresponding graphs.

## Effect of infinite-dimensional model on parameter estimate.

In Figure 2.3 we use $\Delta x=0.005$ and $\Delta t=0.005$ to generate the data and the numerical solution (2.2.3) for the least-squares problem. This removes the infinite-dimensional effect of the partial differential equation given by (2.1.1). However, in Figure 2.4 we use $\Delta x=\Delta t=0.005$ to generate the data and $\Delta x=\Delta t=0.01$ to compute (2.2.3). Thus, in this case the data are not exactly attained by our model even if the noise is removed. We observe that while the estimates in both figures are good, the results in Figures 2.3-2.4 and Table 2.5 suggest that infinite-dimensional effects can lead to a slightly under biased estimator.

Effect of regularization parameter $\alpha$ on parameter estimate.

In Figures 2.5 and 2.6 we change the parameter $\alpha$ while keeping the rest fixed. Clearly, low regularization parameter leads to relatively bad estimates although the estimator in this case seems to be the least biased (see Figure 2.5 and left part of Table 2.6). Increasing the value of $\alpha$ leads to better parameter estimates, but the estimator becomes more under biased (see Figure 2.6 and right part of Table 2.6). If this value is increased more, the


Figure 2.3: $M=10, \alpha=3 e-5$. Each of the grey lines (....) of the left part of the figure denotes a distinct result for a given sample $\left\{\epsilon_{k}\right\}$.

| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :---: | :---: | :---: |
| 0.1 | -0.0778 | -14.4108 | 0.0723 |
| 0.2 | -0.0816 | -8.5015 | 0.1070 |
| 0.3 | -0.0400 | -3.1727 | 0.1012 |
| 0.4 | 0.0110 | 0.7636 | 0.0834 |
| 0.5 | 0.0386 | 2.5745 | 0.0818 |
| 0.6 | 0.0283 | 1.9621 | 0.0868 |
| 0.7 | -0.0124 | -0.9880 | 0.0779 |
| 0.8 | -0.0559 | -5.8206 | 0.0556 |
| 0.9 | -0.0623 | -11.5426 | 0.0280 |


| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :--- | :--- | :--- |
| 0.1 | -0.1236 | -22.8940 | 0.0667 |
| 0.2 | -0.1571 | -16.3628 | 0.1040 |
| 0.3 | -0.1284 | -10.1885 | 0.1141 |
| 0.4 | -0.0785 | -5.4485 | 0.1130 |
| 0.5 | -0.0440 | -2.9329 | 0.1110 |
| 0.6 | -0.0446 | -3.0966 | 0.1049 |
| 0.7 | -0.0754 | -5.9875 | 0.0885 |
| 0.8 | -0.1059 | -11.0334 | 0.0624 |
| 0.9 | -0.0939 | -17.3949 | 0.0323 |

Table 2.5: Left and right tables are statistical results for Figure 2.3 and Figure 2.4, respectively.


Figure 2.4: $M=10, \alpha=3 e-5$. Each of the grey lines (....) of the left part of the figure denotes a distinct result for a given sample $\left\{\epsilon_{k}\right\}$.
estimator is more biased. Also the parameter estimate becomes worse than before. This suggests, not surprisingly, that there is an optimal choice for the parameter $\alpha$ which produces the best results for the parameter estimates.



Figure 2.5: $M=10, \alpha=1 e-5$. Each of the grey lines (....) of the left part of the figure denotes a distinct result for a given sample $\left\{\epsilon_{k}\right\}$.

| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :---: | :---: | :---: |
| 0.1 | -0.1277 | -23.6389 | 0.1206 |
| 0.2 | -0.1648 | -17.1644 | 0.1791 |
| 0.3 | -0.1284 | -10.1938 | 0.1618 |
| 0.4 | -0.0599 | -4.1591 | 0.1221 |
| 0.5 | -0.0072 | -0.4806 | 0.1169 |
| 0.6 | 0.0026 | 0.1788 | 0.1274 |
| 0.7 | -0.0253 | -2.0101 | 0.1126 |
| 0.8 | -0.0631 | -6.5678 | 0.0780 |
| 0.9 | -0.0642 | -11.8944 | 0.0427 |


| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :---: | :---: | :---: |
| 0.1 | -0.1241 | -22.9816 | 0.0506 |
| 0.2 | -0.1621 | -16.8881 | 0.0842 |
| 0.3 | -0.1432 | -11.3627 | 0.1011 |
| 0.4 | -0.1050 | -7.2906 | 0.1078 |
| 0.5 | -0.0791 | -5.2736 | 0.1087 |
| 0.6 | -0.0837 | -5.8139 | 0.1009 |
| 0.7 | -0.1077 | -8.5443 | 0.0847 |
| 0.8 | -0.1288 | -13.4165 | 0.0602 |
| 0.9 | -0.1042 | -19.3027 | 0.0313 |

Table 2.6: Left and right tables are statistical results for Figure 2.5 and Figure 2.6, respectively.

### 2.3.3 $1-D$ linear estimation problem for infinite dimensional parameter space when $N=2$

In this section, we assume $N=2$ and that all the parameters are known except for $\beta^{1}$ and $\beta^{2}$. To estimate $\beta^{1}$ and $\beta^{2}$, we assume that they are of a separable form given by


Figure 2.6: $M=10, \alpha=5 e-5$. Each of the grey lines (....) of the left part of the figure denotes a distinct result for a given sample $\left\{\epsilon_{k}\right\}$.
$\beta^{1}(x, P)=b^{1}(x) \exp (-P)$ and $\beta^{2}(x, P)=b^{2}(x) \exp (-P)$, respectively, where $b^{1}(x)$ and $b^{2}(x)$ are unknown parameters to be identified. To estimate $b^{1}(x)$ and $b^{2}(x)$, we use data which are generated computationally as follows: Let $\gamma^{I, J}=\left\{\begin{array}{ll}1, & I=J \\ 0, & I \neq J\end{array}\right.$ for Figure 2.7 and $\gamma^{I, J}=0.5, I, J=1,2$ for Figure 2.8, $u^{I, 0}(x)=3 \exp \left(-2(x-0.1)^{2}\right)$, and for the parameters $g^{I}, m^{I}$ and $\beta^{I}$ we use the following choice of functions:

$$
\begin{aligned}
& g^{1}=2(1-x) \exp (-0.8 P), \quad g^{2}=(1-x)(1+2 P) \exp (-P) \\
& m^{1}=\exp \left(2(x-0.4)^{2}\right) \exp (0.2 P), \quad m^{2}=\exp \left(2(x-0.4)^{2}\right) \exp (0.2 P) \\
& \beta^{1}=6(1-x) x \exp (-P), \quad \beta^{2}=6(1-x) x \exp \left(-5(x-0.5)^{2}\right) \exp (-P)
\end{aligned}
$$

and solve (2.1.1) for $U_{\Delta x, \Delta t}^{I}(x, t), I=1,2$. We set the data $Z_{I, k}=\left(1+\varepsilon_{I, k}\right) \int_{0}^{1} U_{\Delta x, \Delta t}^{I}\left(x, t_{k}\right) d x$, $I=1,2$ for Figure 2.7 and $Z_{k}=\left(1+\varepsilon_{k}\right) \sum_{I=1}^{2} \int_{0}^{1} U_{\Delta x, \Delta t}^{I}\left(x, t_{k}\right) d x$ for Figure 2.8, where $\varepsilon_{I, k}$ and $\varepsilon_{k}$ both are the random sample from a normal random number generator with mean zero and standard deviation $\sigma=0.02$.

We choose the parameter space $Q=\mathcal{D} \times \mathcal{D}$. Clearly, $Q$ is compact in $\mathcal{C}[0,1] \times \mathcal{C}[0,1]$. We approximate the infinite dimensional parameter space as follows: For $M_{1}, M_{2}$ positive
integers and any $\left(b_{1}, b_{2}\right) \in Q$, we set

$$
\left(\mathcal{I}_{M_{J}} b^{J}\right)(x)=\sum_{i=1}^{M_{J}-1} b^{J}\left(\frac{i}{M_{J}}\right) \phi_{M_{J}}^{i}(x ; 0,1), \quad J=1,2 .
$$

Clearly, $\lim _{M_{J} \rightarrow \infty} \mathcal{I}_{M_{J}} b^{J}=b^{J}$ in $\mathcal{C}[0,1]$, uniformly in $b^{J}, J=1,2$. Hence, if $b_{M_{J}}^{J} \in Q_{M_{J}}=$ $I_{M_{J}}(Q)$ is given by

$$
b_{M_{J}}^{J}(x)=\sum_{i=1}^{M_{J}-1} \lambda_{M_{J}}^{J, i} \phi_{M_{J}}^{i}(x ; 0,1), \quad J=1,2,
$$

then the solution of our finite dimensional identification problem involves identifying the $M_{1}+M_{2}-2$ coefficients $\left\{\lambda_{M_{J}}^{J, i}\right\}_{i=1, J=1}^{M_{J}-1,2}$ from a compact subset of $\mathbb{R}_{+}^{M_{1}+M_{2}-2}$ so as to minimize the least-squares cost functional (2.2.4) or (2.2.5).

In order to indirectly implement the compactness constraints of $Q$, we still use the regularized least-squares cost functional. For Figure 2.7 we use the form

$$
\begin{aligned}
\mathcal{J}_{\Delta x, \Delta t}(q)= & \sum_{I=1}^{2} \sum_{k=1}^{m}\left|\log \left(\int_{0}^{1} U_{\Delta x, \Delta t}^{I}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{I, k}+1\right)\right|^{2} \\
& +\sum_{I=1}^{2} \alpha_{I} \int_{0}^{1}\left|\frac{d}{d x} b_{M_{I}}^{I}(x)\right|^{2} d x
\end{aligned}
$$

and for Figure 2.8 we use the form

$$
\begin{aligned}
\mathcal{J}_{\Delta x, \Delta t}(q)= & \sum_{k=1}^{m}\left|\log \left(\sum_{I=1}^{2} \int_{0}^{1} U_{\Delta x, \Delta t}^{I}\left(x, t_{k} ; q\right) d x+1\right)-\log \left(Z_{k}+1\right)\right|^{2} \\
& +\sum_{I=1}^{2} \alpha_{I} \int_{0}^{1}\left|\frac{d}{d x} b_{M_{I}}^{I}(x)\right|^{2} d x
\end{aligned}
$$

where $\alpha_{I}>0, I=1,2$ are the regularization parameters and $m=100$ for Figures 2.7 and 2.8.

In the rest of our simulations we use $\Delta x=\Delta t=0.005$ to generate the data and $\Delta x=\Delta t=0.01$ to solve the least-squares. Thus, in these cases the data are not exactly
attained by our model even if the noise is removed.
The upper-left part and the lower-left part of the following two figures represent the $S(=1000)$ numerical results of the estimated parameters $b_{M_{1}}^{1}(x)$ and $b_{M_{2}}^{2}(x)$ versus the exact parameters $b^{1}(x)$ and $b^{2}(x)$, respectively. The upper-right part and the lower right part represent the figures of the corresponding $95 \%$ confidence interval (dashed line) versus the exact $b^{1}(x)$ and $b^{2}(x)$ (solid line), respectively. The tables provide statistical results for the corresponding graphs.

Note that the results in Figure 2.7 and Table 2.7 are slightly better than those in Figure 2.8 and Table 2.8. This is expected since in Figure 2.7 we are sampling data for each of the two populations, which provides more information than sampling the sum of the two populations only, as is the case in Figure 2.8. Also note that in both of these figures we let $M=M_{1}=M_{2}=10$.


Figure 2.7: $M=10, \alpha_{1}=5 e-5, \alpha_{2}=5 e-5$. Each of the grey lines (....) of the left part of the figure denotes a distinct result for a given sample $\left\{\epsilon_{I, k}\right\}$.

| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :---: | :---: | :---: |
| 0.1 | -0.0187 | -3.4717 | 0.0880 |
| 0.2 | -0.0004 | -0.0447 | 0.1276 |
| 0.3 | 0.0334 | 2.6514 | 0.1053 |
| 0.4 | 0.0562 | 3.9007 | 0.0493 |
| 0.5 | 0.0449 | 2.9941 | 0.0548 |
| 0.6 | -0.0040 | -0.2805 | 0.0860 |
| 0.7 | -0.0683 | -5.4239 | 0.0836 |
| 0.8 | -0.1101 | -11.4644 | 0.0576 |
| 0.9 | -0.0929 | -17.2091 | 0.0272 |


| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :---: | :---: | :---: |
| 0.1 | 0.1684 | 69.4034 | 0.0959 |
| 0.2 | 0.1628 | 26.5887 | 0.1528 |
| 0.3 | 0.0487 | 4.7244 | 0.1483 |
| 0.4 | -0.0728 | -5.3114 | 0.0946 |
| 0.5 | -0.1134 | -7.5604 | 0.0464 |
| 0.6 | -0.0437 | -3.1871 | 0.0860 |
| 0.7 | 0.0931 | 9.0282 | 0.1053 |
| 0.8 | 0.2039 | 33.3052 | 0.0819 |
| 0.9 | 0.1954 | 80.5164 | 0.0402 |

Table 2.7: Left and right tables are statistical results of $b^{1}(x)$ and $b^{2}(x)$ for Figure 2.7, respectively.


Figure 2.8: $M=10, \alpha_{1}=5 e-5, \alpha_{2}=5 e-5$. Each of the grey lines (....) of the left part of the figure denotes a distinct result for a given sample $\left\{\epsilon_{k}\right\}$.

| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :---: | :---: | :---: |
| 0.1 | -0.0687 | -12.7279 | 0.0765 |
| 0.2 | -0.0790 | -8.2334 | 0.1096 |
| 0.3 | -0.0572 | -4.5419 | 0.0891 |
| 0.4 | -0.0402 | -2.7920 | 0.0435 |
| 0.5 | -0.0537 | -3.5800 | 0.0588 |
| 0.6 | -0.0980 | -6.8075 | 0.0871 |
| 0.7 | -0.1490 | -11.8273 | 0.0846 |
| 0.8 | -0.1694 | -17.6443 | 0.0596 |
| 0.9 | -0.1255 | -23.2483 | 0.0296 |


| $x$ | $A B(x)$ | $R A B(x)$ | $S E(x)$ |
| :---: | :---: | :---: | :---: |
| 0.1 | 0.1926 | 79.3867 | 0.1066 |
| 0.2 | 0.2106 | 34.4018 | 0.1757 |
| 0.3 | 0.1187 | 11.5041 | 0.1784 |
| 0.4 | 0.0178 | 1.2960 | 0.1208 |
| 0.5 | -0.0069 | -0.4598 | 0.0549 |
| 0.6 | 0.0665 | 4.8565 | 0.0915 |
| 0.7 | 0.1889 | 18.3112 | 0.1157 |
| 0.8 | 0.2704 | 44.1765 | 0.0915 |
| 0.9 | 0.2239 | 92.2680 | 0.0459 |

Table 2.8: Left and right tables are statistical results of $b^{1}(x)$ and $b^{2}(x)$ for Figure 2.8, respectively.

### 2.4 Concluding Remarks

In this paper we have developed a numerical technique for identifying unknown parameters in a general size-structured population model. A main focus of this chapter is on the statistical study of the parameter estimation technique. This was done via thousands of numerical experiments.

Several conclusions can be drawn from our studies. 1) The method discussed above seems to perform well and produce good confidence intervals for the parameters. 2) When the infinite dimensional effects of the model and the parameter space are removed, the resulting numerical and statistical values suggest that the least-squares technique produces very good unbiased parameter estimates. 3) The type of numerical scheme used for approximating the infinite-dimensional model as well as the parameter space may influence the bias in the parameter estimation technique. 4) The commonly used regularization term is crucial for enforcing compactness and obtaining better estimates. However, it may also introduce more bias in the estimator.

We note in closing that the system (2.1.1) investigated in this paper is a special case of the measure dependent aggregate dynamics problems formulated in [6] wherein individual (uncoupled) dynamics are not available. Inverse problems for such systems have been investigated in a number of applications including cellular level HIV modelling [7],
hysteresis in viscoelastic materials [8, 9], shear waves in biotissue [10], and electromagnetic interrogation in complex materials [11]. In a more general formulation (currently under investigation by the authors), one has a probability distribution $F$ of individual parameters $q(x, P)=q=(g, m, \beta, C)$ on an admissible set $Q$. The system (2.1.1) is replaced by a continuum of systems for $u(x, t ; q(x, P))$ with the total population $P(t ; F)$ given by

$$
P(t ; F)=\int_{Q}\left[\int_{0}^{L} u(x, t ; q) d x\right] d F(q)=\int_{Q}\left[\int_{0}^{L} u(x, t ; q) d x\right] f(q) d q
$$

the latter equality holding if $F$ has a density $f$. The aggregate dynamics for $u$ depend explicitly on $F$ through the dependence of the individual rate parameters $(g, m, \beta, C)$ on the total population $P$.

If $F$ is a discrete measure with $N$ atoms at $q^{J}$ of mass $f_{J}$, then we have

$$
P(t ; F)=\sum_{J=1}^{N} f_{J} \int_{0}^{L} u\left(x, t ; q^{J}\right) d x .
$$

Moreover, if $F$ is uniformly and discretely distributed $\left(f_{J}=\frac{1}{N}\right)$, this becomes

$$
P(t ; F)=\frac{1}{N} \sum_{J=1}^{N} \int_{0}^{L} u\left(x, t ; q^{J}\right) d x
$$

which is simply a scaled (by $\frac{1}{N}$ ) version of (2.1.2). Of course, even in this simple case, the system does not decouple. (i.e., individual dynamics are not available). This will be the case anytime the individual parameters for subpopulations depend on the total population. It is also clear that inverse problems with such measure dependent dynamics are a generalized version of the estimation problems discussed in the statistical literature in the context of hierarchial modelling [36].

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## Chapter 3

## On a Nonlinear Size-Structured Phytoplankton-Zooplankton Aggregation Model

We consider a coupled system of nonlinear size-structured phytoplankton population and zooplankton population. We develop a comparison principle and construct monotone sequences to show the existence of the solution. The uniqueness of the solution is also established.

### 3.1 Introduction

In this chapter, we consider the following initial boundary value problem that describes the dynamics of coupled size-structured phytoplankton-zooplankton system

$$
\begin{align*}
& u_{t}+\left(g_{1}(x, t) u(x, t)\right)_{x}+m_{1}\left(x, t, \varphi^{u}, \varphi^{z}\right) u(x, t)=\frac{1}{2} \int_{0}^{x} \beta(x-y, y) u(x-y, t) u(y, t) d y \\
& -\int_{0}^{\infty} \beta(x, y) u(x, t) u(y, t) d y \quad 0<x<\infty, 0<t<T \\
& z_{t}+\left(g_{2}(x, t) z(x, t)\right)_{x}+m_{2}\left(x, t, \varphi^{u}, \varphi^{z}\right) z(x, t)=0 \quad 0<x<\infty, 0<t<T \\
& g_{1}(0, t) u(0, t)=\int_{0}^{\infty} \gamma_{1}\left(y, t, \varphi^{u}, \varphi^{z}\right) u(y, t) d y \quad 0<t<T \\
& g_{2}(0, t) z(0, t)=\int_{0}^{\infty} \gamma_{2}\left(y, t, \varphi^{u}, \varphi^{z}\right) z(y, t) d y \quad 0<t<T \\
& u(x, 0)=u_{0}(x) \quad 0 \leq x<\infty \\
& z(x, 0)=z_{0}(x) \quad 0 \leq x<\infty, \tag{3.1.1}
\end{align*}
$$

where

$$
\varphi^{u}=\int_{0}^{\infty} u(x, t) d x \quad \text { and } \quad \varphi^{z}=\int_{0}^{\infty} z(x, t) d x
$$

Here $u(x, t)$ and $z(x, t)$ are the density of aggregates having size $x$ at time $t$ of phytoplankton population and zooplankton population, respectively. The function $\beta(x, y)$ is the rate at which particles of size $x$ coagulate with particles of size $y$. The functions $g_{1}$ and $g_{2}$ denote the growth rate of an aggregate in the phytoplankton population and zooplankton population, respectively. The functions $m_{1}$ and $m_{2}$ denote the mortality rate of an aggregate in the phytoplankton population and zooplankton population, respectively. The function $\gamma_{1}$ and $\gamma_{2}$ are the number of single cells that fall off an aggregate of size $x$ and join the single cell population of phytoplankton population and zooplankton population, respectively. The first integral term on the right side of (3.1.1) expresses the rate at which collisions occur to form new particles in the size interval $(x, x+d x)$, while the second term represents the rate at which such collisions cause these particles to be lost from the same interval. The third and forth integral term represent the addition of new born single cells to the single cell phytoplankton population and zooplankton population,respectively.

There is a growing literature on investigating the aggregation model of phytoplankton cells. In [2], the inverse problem of identifying the compactly supported coagulation kernel with $g \equiv 0, m \equiv 0$ and $\gamma \equiv 0$ was studied. In [3] the inverse problem of identifying parameters in the model discussed in [5] from observed data was investigated. In addition, four common methods are used in the literature to establish the existence-uniqueness for its certain case. One is the semigroups of linear operators theoretic approach, which is used in $[9,10]$ to deal with the case of a bounded coagulation kernel with $g \equiv 0, m \equiv 0$ and $\gamma \equiv 0$, and is also used in [5] to deal with the case of a bounded domain and a compactly supported coagulation kernel with $m \equiv 0$ and $\gamma=\gamma(y)$. The second approach
is evolution operators theory, which is used in [11] to extend those results in [9, 10] to a time-dependent coagulation kernel. The third approach is finite difference scheme used in [4]. The last approach is the monotone method, which is used in [1] for the case of infinite domain for the particle size, $\gamma=\gamma(y, t)$ and $m \equiv 0$.

The goal of this chapter is to extend the results in [1] for a coupled size-structured phytoplankton-zooplankton system with the parameters $m$ and $\gamma$ both being depending on the total population. Techniques in the spirit of those in [1] are used to establish the comparison principle and obtain the existence of the solution. To our knowledge, results on the existence and uniqueness for this kind of coupled system with quasilinear case given in (3.1.1) are not available in the literature.

The remainder of this chapter is organized as follows. In Section 3.2, we define a pair of coupled upper and lower solutions and establish a comparison principle. In Section 3.3, we construct two monotone sequences of upper and lower solutions, and then show the existence of the solution of problem (3.1.1). In Section 3.4, we establish the uniqueness of the solution, and we also show that this local solution is a global one.

### 3.2 Comparison Principle

For convenience, we use $\left\|\|_{\infty}\right.$ to denote the supreme of a function in its domain throughout this chapter. In order to carry out our programme, the following conditions will be imposed:
(H1) $u_{0}(x)$ and $z_{0}(x)$ are non-negative function on $[0, \infty)$, and $u_{0}, z_{0} \in \mathbf{L}^{1}(0, \infty) \cap$ $\mathbf{L}^{\infty}(0, \infty)$.
(H2) $\beta(x, y)$ is a continuous, non-negative function on $[0, \infty) \times[0, \infty)$ with $\|\beta\|_{\infty}<\infty$.
(H3) $m_{i}\left(x, t, \varphi^{u}, \varphi^{z}\right)$ is a bounded and non-negative function on $[0, \infty) \times[0, T]$. We further assumed that $m_{i}\left(x, t, \varphi^{u}, \varphi^{z}\right)$ is continuously differentiable with respect to
$\varphi^{u}$ and $\varphi^{z},\left\|m_{i \varphi^{u}}\right\|_{\infty}<\infty$, and $\left\|m_{i \varphi^{z}}\right\|_{\infty}<\infty$ for $i=1,2$. We also assumed that $m_{1 \varphi^{u}} \geq 0, m_{1 \varphi^{z}} \geq 0, m_{2 \varphi^{u}} \leq 0$ and $m_{2 \varphi^{z}} \geq 0$.
(H4) $\gamma_{i}\left(x, t, \varphi^{u}, \varphi^{z}\right)$ is a bounded and non-negative function on $[0, \infty) \times[0, T]$. We further assumed that $\gamma_{i}\left(x, t, \varphi^{u}, \varphi^{z}\right)$ is continuously differentiable with respect to $\varphi^{u}$ and $\varphi^{z},\left\|\gamma_{i \varphi^{u}}\right\|_{\infty}<\infty$, and $\left\|\gamma_{i \varphi^{z}}\right\|_{\infty}<\infty$ for $i=1,2$. We also assumed that $\gamma_{1 \varphi^{u}} \leq 0$, $\gamma_{1 \varphi^{z}} \leq 0, \gamma_{2 \varphi^{u}} \geq 0$ and $\gamma_{2 \varphi^{z}} \leq 0$.
(H5) $g_{i}(x, t)$ is continuously differentiable on $(0, \infty) \times(0, T)$ with $\left\|g_{1 x}\right\|_{\infty}<\infty$ and $\left\|g_{2 x}\right\|_{\infty}<\infty$. Furthermore, $g_{i}(x, t)>0$ for $(x, t) \in[0, \infty) \times[0, T]$ and $\lim _{x \rightarrow \infty} g_{i}(x, t)=$ 0 for $t \in[0, T]$.

For simplicity, let $\mathbb{D}_{T}=(0, \infty) \times(0, T)$ and $\mathbf{C}_{0, r}^{1}\left(\mathbb{D}_{T}\right)=\left\{\psi \in \mathbf{C}^{1}\left(\mathbb{D}_{T}\right)\right.$ : there exists a constant $x_{\psi} \in(0, \infty)$ such that $\psi \equiv 0$ for $\left.x \geq x_{\psi}\right\}$. We then introduce the definition of coupled upper and lower solutions of (3.1.1) as follows:

Definition 3.2.1. A pair of functions $(\bar{u}(x, t), \bar{z}(x, t))$ and $(\underline{u}(x, t), \underline{z}(x, t))$ are called an upper and lower solution of (3.1.1) on $\mathbb{D}_{T}$, respectively, if all the following hold:
(i) $\bar{u}, \underline{u}, \bar{z}, \underline{z} \in \mathbf{L}^{\infty}\left((0, T) ; \mathbf{L}^{1}(0, \infty)\right)$.
(ii) $\bar{u}(x, 0) \geq u_{0}(x) \geq \underline{u}(x, 0), \bar{z}(x, 0) \geq z_{0}(x) \geq \underline{z}(x, 0)$ a.e. in $(0, \infty)$.
(iii) For every $t \in(0, T)$ and every non-negative $\xi, \eta \in \mathbf{C}_{0, r}^{1}\left(\mathbb{D}_{T}\right)$, we have

$$
\begin{align*}
& \int_{0}^{\infty} \bar{u}(x, t) \xi(x, t) d x \\
\geq & \int_{0}^{\infty} \bar{u}(x, 0) \xi(x, 0) d x+\int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{\underline{u}}, \varphi^{\underline{z}}\right) \bar{u}(x, s) d x d s \\
& +\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g_{1}(x, s) \xi_{x}(x, s)\right] \bar{u}(x, s) d x d s  \tag{3.2.1}\\
& +\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y) \bar{u}(x-y, s) \bar{u}(y, s) d y d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty} \xi(x, s)\left[\int_{0}^{\infty} \beta(x, y) \underline{u}(y, s) d y+m_{1}\left(x, s, \varphi^{\underline{u}}, \varphi^{\underline{z}}\right)\right] \bar{u}(x, s) d x d s
\end{align*}
$$

$$
\begin{align*}
& \int_{0}^{\infty} \underline{u}(x, t) \xi(x, t) d x \\
& \leq \int_{0}^{\infty} \underline{u}(x, 0) \xi(x, 0) d x+\int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right) \underline{u}(x, s) d x d s \\
&+\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g_{1}(x, s) \xi_{x}(x, s)\right] \underline{u}(x, s) d x d s  \tag{3.2.2}\\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y) \underline{u}(x-y, s) \underline{u}(y, s) d y d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} \xi(x, s)\left[\int_{0}^{\infty} \beta(x, y) \bar{u}(y, s) d y+m_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right)\right] \underline{u}(x, s) d x d s, \\
& \\
& \quad \int_{0}^{\infty} \bar{z}(x, t) \eta(x, t) d x  \tag{3.2.3}\\
& \geq \int_{0}^{\infty} \bar{z}(x, 0) \eta(x, 0) d x+\int_{0}^{t} \eta(0, s) \int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{\bar{u}}, \varphi^{\underline{z}}\right) \bar{z}(x, s) d x d s \\
&+\int_{0}^{t} \int_{0}^{\infty}\left[\eta_{s}(x, s)+g_{2}(x, s) \eta_{x}(x, s)\right] \bar{z}(x, s) d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} m_{2}\left(x, s, \varphi^{\bar{u}}, \varphi^{z}\right) \bar{z}(x, s) \eta(x, s) d x d s,
\end{align*}
$$

and

$$
\begin{align*}
& \int_{0}^{\infty} \underline{z}(x, t) \eta(x, t) d x \\
\leq & \int_{0}^{\infty} \underline{z}(x, 0) \eta(x, 0) d x+\int_{0}^{t} \eta(0, s) \int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right) \underline{z}(x, s) d x d s  \tag{3.2.4}\\
& +\int_{0}^{t} \int_{0}^{\infty}\left[\eta_{s}(x, s)+g_{2}(x, s) \eta_{x}(x, s)\right] \underline{z}(x, s) d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty} m_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right) \underline{z}(x, s) \eta(x, s) d x d s .
\end{align*}
$$

Definition 3.2.2. $(u(x, t), z(x, t))$ is called a solution of (3.1.1) on $\mathbb{D}_{T}$ if $(u, z)$ satisfies (3.2.1) and (3.2.3) with " $\geq$ " replaced by " $=$ ", and $(\bar{u}, \bar{z})$ and $(\underline{u}, \underline{z})$ are both replaced by $(u, z)$.

Theorem 3.2.1. Suppose that (H1)-(H5) hold. Let $(\bar{u}, \bar{z})$ and $(\underline{u}, \underline{z})$ be the non-negative upper solution and non-negative lower solution of (3.1.1), respectively. Then $\bar{u} \geq \underline{u}$ and $\bar{z} \geq \underline{z}$ a.e. in $\mathbb{D}_{T}$.

Proof. Let $v=\underline{u}-\bar{u}$ and $w=\underline{z}-\bar{z}$. Choose non-negative functions $\xi$ and $\eta \in \mathbf{C}_{0, r}^{1}((0, n) \times$ $(0, T))$. Then $v$ and $w$ satisfy $v(x, 0) \leq 0$ and $w(x, 0) \leq 0$ a.e. in $[0, \infty)$. By (3.2.1) and
(3.2.2), we find

$$
\begin{aligned}
& \int_{0}^{\infty} \quad v(x, t) \xi(x, t) d x \leq \int_{0}^{\infty} v(x, 0) \xi(x, 0) d x \\
& \quad+\int_{0}^{t} \xi(0, s) \int_{0}^{\infty}\left[\gamma_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right) v(x, s)+\left(\gamma_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right)-\gamma_{1}\left(x, s, \varphi^{\underline{u}}, \varphi^{\underline{z}}\right)\right) \bar{u}(x, s)\right] d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g_{1}(x, s) \xi_{x}(x, s)\right] v(x, s) d x d s \\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y)[\underline{u}(x-y, s) v(y, s)+v(x-y, s) \bar{u}(y, s)] d y d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) v(x, s) \int_{0}^{\infty} \beta(x, y) \bar{u}(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \bar{u}(x, s) \int_{0}^{\infty} \beta(x, y) v(y, s) d y d x d s-\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) m_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right) v(x, s) d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} \xi(x, s)\left[m_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right)-m_{1}\left(x, s, \varphi^{\underline{u}}, \varphi^{\underline{z}}\right)\right] \bar{u}(x, s) d x d s .
\end{aligned}
$$

Rewriting some terms in the right side of the above equation, we obtain

$$
\begin{aligned}
& \frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y)[\underline{u}(x-y, s) v(y, s)+v(x-y, s) \bar{u}(y, s)] d y d x d s \\
= & \frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \int_{y}^{\infty} \xi(x, s) \beta(x-y, y)[\underline{u}(x-y, s) v(y, s)+v(x-y, s) \bar{u}(y, s)] d x d y d s \\
= & \frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} v(y, s) \int_{0}^{\infty} \xi(y+z, s) \beta(z, y) \underline{u}(z, s) d z d y d s \\
& +\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \bar{u}(y, s) \int_{0}^{\infty} \xi(y+z, s) \beta(z, y) v(z, s) d z d y d s,
\end{aligned}
$$

$$
\begin{aligned}
& -\int_{0}^{t} \int_{0}^{\infty} \xi(x, s)\left[m_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right)-m_{1}\left(x, s, \varphi^{\underline{u}}, \varphi^{\underline{z}}\right)\right] \bar{u}(x, s) d x d s \\
= & -\int_{0}^{t} \int_{0}^{\infty} \xi(x, s)\left[m_{1 \varphi^{u}}\left(x, s, \theta_{u_{m}}, \varphi^{\bar{z}}\right)\left(\varphi^{\bar{u}}-\varphi^{\underline{u}}\right)+m_{1 \varphi^{z}}\left(x, s, \varphi^{\underline{u}}, \theta_{z_{m}}\right)\left(\varphi^{\bar{z}}-\varphi^{\underline{z}}\right)\right] \bar{u}(x, s) d x d s \\
= & \int_{0}^{t} \int_{0}^{\infty} \bar{u}(x, s) \xi(x, s) m_{1 \varphi^{u}}\left(x, s, \theta_{u_{m}}, \varphi^{\bar{z}}\right) \int_{0}^{\infty} v(y, s) d y d x d s \\
& +\int_{0}^{t} \int_{0}^{\infty} \bar{u}(x, s) \xi(x, s) m_{1 \varphi^{z}}\left(x, s, \varphi^{\underline{u}}, \theta_{z_{m}}\right) \int_{0}^{\infty} w(y, s) d y d x d s,
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{0}^{t} \xi(0, s) \int_{0}^{\infty}\left[\gamma_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right)-\gamma_{1}\left(x, s, \varphi^{\underline{u}}, \varphi^{\underline{z}}\right)\right] \bar{u}(x, s) d x d s \\
= & \int_{0}^{t} \xi(0, s) \int_{0}^{\infty}\left[\gamma_{1 \varphi^{u}}\left(x, s, \theta_{u_{\gamma}}, \varphi^{\bar{z}}\right)\left(\varphi^{\bar{u}}-\varphi^{\underline{u}}\right)+\gamma_{1 \varphi^{z}}\left(x, s, \varphi^{\underline{u}}, \theta_{z_{\gamma}}\right)\left(\varphi^{\bar{z}}-\varphi^{\underline{z}}\right)\right] \bar{u}(x, s) d x d s \\
= & -\int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{1 \varphi^{u}}\left(x, s, \theta_{u_{\gamma}}, \varphi^{\bar{z}}\right) \bar{u}(x, s) \int_{0}^{\infty} v(y, s) d y d x d s \\
& -\int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{1 \varphi^{z}}\left(x, s, \varphi^{\underline{u}}, \theta_{z_{\gamma}}\right) \bar{u}(x, s) \int_{0}^{\infty} w(y, s) d y d x d s,
\end{aligned}
$$

where $\theta_{u_{m}}$ and $\theta_{u_{\gamma}}$ are both between $\varphi^{\underline{u}}$ and $\varphi^{\bar{u}}$, and $\theta_{z_{m}} \theta_{z_{\gamma}}$ are both between $\varphi^{\underline{z}}$ and $\varphi^{\bar{z}}$. Note that $\xi(x, 0) \geq 0$, and $v(x, 0) \leq 0$ a.e. in $(0, \infty)$, we have

$$
\begin{align*}
& \int_{0}^{\infty} v(x, t) \xi(x, t) d x \leq \int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right) v(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g_{1}(x, s) \xi_{x}(x, s)\right] v(x, s) d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} \xi(x, s)\left[\int_{0}^{\infty} \beta(x, y) \bar{u}(y, s) d y+m_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right)\right] v(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} \bar{u}(x, s) \int_{0}^{\infty} \xi(x, s) \beta(x, y) v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} \bar{u}(x, s)\left[-\xi(0, s) \gamma_{1 \varphi^{u}}\left(x, s, \theta_{u_{\gamma}}, \varphi^{\bar{z}}\right)+\xi(x, s) m_{1 \varphi^{u}}\left(x, s, \theta_{u_{m}}, \varphi^{\bar{z}}\right)\right] \int_{0}^{\infty} v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} \bar{u}(x, s)\left[-\xi(0, s) \gamma_{1 \varphi^{z}}\left(x, s, \varphi^{\underline{u}}, \theta_{z_{\gamma}}\right)+\xi(x, s) m_{1 \varphi^{z}}\left(x, s, \varphi^{\underline{u}}, \theta_{z_{m}}\right)\right] \int_{0}^{\infty} w(y, s) d y d x d s \\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} v(y, s) \int_{0}^{\infty} \xi(y+z, s) \beta(z, y) \underline{u}(z, s) d z d y d s \\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \bar{u}(y, s) \int_{0}^{\infty} \xi(y+z, s) \beta(z, y) v(z, s) d z d y d s . \tag{3.2.5}
\end{align*}
$$

By (3.2.3) and (3.2.4), we find

$$
\begin{aligned}
& \int_{0}^{\infty} \quad w(x, t) \eta(x, t) d x \leq \int_{0}^{\infty} w(x, 0) \eta(x, 0) d x \\
& \quad+\int_{0}^{t} \eta(0, s) \int_{0}^{\infty}\left[\gamma_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right) w(x, s)+\left(\gamma_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right)-\gamma_{2}\left(x, s, \varphi^{\bar{u}}, \varphi^{\underline{z}}\right)\right) \bar{z}(x, s)\right] d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\eta_{s}(x, s)+g_{2}(x, s) \eta_{x}(x, s)\right] w(x, s) d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} \eta(x, s) m_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right) w(x, s) d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} \eta(x, s)\left[m_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right)-m_{2}\left(x, s, \varphi^{\bar{u}}, \varphi^{\underline{z}}\right)\right] \bar{z}(x, s) d x d s .
\end{aligned}
$$

Note that $\eta(x, 0) \geq 0$, and $w(x, 0) \leq 0$ a.e. in $(0, \infty)$, we have

$$
\begin{align*}
& \int_{0}^{\infty} w(x, t) \eta(x, t) d x \leq \int_{0}^{t} \eta(0, s) \int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right) w(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\eta(0, s) \gamma_{2 \varphi^{u}}\left(x, s, \theta_{1}, \varphi^{\bar{z}}\right)-\eta(x, s) m_{2 \varphi^{u}}\left(x, s, \theta_{2}, \varphi^{\bar{z}}\right)\right] \bar{z}(x, s) \int_{0}^{\infty} v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[-\eta(0, s) \gamma_{2 \varphi^{z}}\left(x, s, \varphi^{\bar{u}}, \theta_{3}\right)+\eta(x, s) m_{2 \varphi^{z}}\left(x, s, \varphi^{\bar{u}}, \theta_{4}\right)\right] \bar{z}(x, s) \int_{0}^{\infty} w(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\eta_{s}(x, s)+g_{2}(x, s) \eta_{x}(x, s)\right] w(x, s) d x d s-\int_{0}^{t} \int_{0}^{\infty} \eta(x, s) m_{2}\left(x, s, \varphi^{u}, \varphi^{\bar{z}}\right) w(x, s) d x d s, \tag{3.2.6}
\end{align*}
$$

where $\theta_{1}$ and $\theta_{2}$ are both between $\varphi^{\underline{u}}$ and $\varphi^{\bar{u}}$, and $\theta_{3} \theta_{4}$ are both between $\varphi^{\underline{\underline{z}}}$ and $\varphi^{\bar{z}}$. Let $\xi(x, t)=e^{\lambda_{1} t} \zeta(x, t)$, where $\zeta \in \mathbf{C}_{0, r}^{1}((0, n) \times(0, T))$ and $\lambda_{1}$ is chosen so that $\lambda_{1}-$
$\int_{0}^{\infty} \beta(x, y) \bar{u}(y, s) d y-m_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right) \geq 0$ on $\mathbb{D}_{T}$. Then by (3.2.5), we obtain

$$
\begin{align*}
& e^{\lambda_{1} t} \int_{0}^{\infty} v(x, t) \zeta(x, t) d x \leq \int_{0}^{t} e^{\lambda_{1} s} \zeta(0, s) \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right) v(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\zeta_{s}(x, s)+g_{1}(x, s) \zeta_{x}(x, s)\right] e^{\lambda_{1} s} v(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} \zeta(x, s) v(x, s)\left[\lambda_{1}-\int_{0}^{\infty} \beta(x, y) \bar{u}(y, s) d y-m_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right)\right] d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} \bar{u}(x, s) \int_{0}^{\infty} \zeta(x, s) \beta(x, y) v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} \bar{u}(x, s)\left[-\zeta(0, s) \gamma_{1 \varphi^{u}}\left(x, s, \theta_{u_{\gamma}}, \varphi^{\bar{z}}\right)+\zeta(x, s) m_{1 \varphi^{u}}\left(x, s, \theta_{u_{m}}, \varphi^{\bar{z}}\right)\right] \int_{0}^{\infty} v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} \bar{u}(x, s)\left[-\zeta(0, s) \gamma_{1 \varphi^{z}}\left(x, s, \varphi^{\underline{u}}, \theta_{z_{\gamma}}\right)+\zeta(x, s) m_{1 \varphi^{z}}\left(x, s, \varphi^{\underline{u}}, \theta_{z_{m}}\right)\right] \int_{0}^{\infty} w(y, s) d y d x d s \\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} v(y, s) \int_{0}^{\infty} \zeta(y+z, s) \beta(z, y) \underline{u}(z, s) d z d y d s \\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} \bar{u}(y, s) \int_{0}^{\infty} \zeta(y+z, s) \beta(z, y) v(z, s) d z d y d s . \tag{3.2.7}
\end{align*}
$$

Let $\eta(x, t)=e^{\lambda_{2} t} \rho(x, t)$, where $\rho \in \mathbf{C}_{0, r}^{1}((0, n) \times(0, T))$ and $\lambda_{2}$ is chosen so that $\lambda_{2}-$ $m_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right) \geq 0$ on $\mathbb{D}_{T}$. Then by (3.2.6), we find

$$
\begin{align*}
& e^{\lambda_{2} t} \int_{0}^{\infty} w(x, t) \rho(x, t) d x \leq \int_{0}^{t} e^{\lambda_{2} s} \rho(0, s) \int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{u}, \varphi^{\bar{z}}\right) w(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\rho(0, s) \gamma_{2 \varphi^{u}}\left(x, s, \theta_{1}, \varphi^{\bar{z}}\right)-\rho(x, s) m_{2 \varphi^{u}}\left(x, s, \theta_{2}, \varphi^{\bar{z}}\right)\right] e^{\lambda_{2} s} \bar{z}(x, s) \int_{0}^{\infty} v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[-\rho(0, s) \gamma_{2 \varphi^{z}}\left(x, s, \varphi^{\bar{u}}, \theta_{3}\right)+\rho(x, s) m_{2 \varphi^{z}}\left(x, s, \varphi^{\bar{u}}, \theta_{4}\right)\right] e^{\lambda_{2} s} \bar{z}(x, s) \int_{0}^{\infty} w(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\rho_{s}(x, s)+g_{2}(x, s) \rho_{x}(x, s)\right] e^{\lambda_{2} s} w(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{2} s} \rho(x, s)\left[\lambda_{2}-m_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right)\right] w(x, s) d x d s . \tag{3.2.8}
\end{align*}
$$

We now set up two backward problems as follows:

$$
\begin{array}{ll}
\zeta_{s}(x, s)+g_{1}(x, s) \zeta_{x}(x, s)=0, & 0<s<t, 0<x<n \\
\zeta(n, s)=0, & 0<s<t \\
\zeta(x, t)=\chi_{1}(x), & 0 \leq x \leq n
\end{array}
$$

and

$$
\begin{array}{ll}
\rho_{s}(x, s)+g_{2}(x, s) \rho_{x}(x, s)=0, & 0<s<t, 0<x<n \\
\rho(n, s)=0, & 0<s<t \\
\rho(x, t)=\chi_{2}(x), & 0 \leq x \leq n .
\end{array}
$$

Here $\chi_{i} \in \mathbf{C}_{0}^{\infty}(0, n)$ and $0 \leq \chi_{i} \leq n, i=1,2$. The existence of $\zeta(x, s)$ and $\rho(x, s)$ can be easily shown. Note that the initial and boundary conditions of $\zeta(x, s)$ and $\rho(x, s)$, we have $0 \leq \zeta(x, s) \leq 1$ and $0 \leq \rho(x, s) \leq 1$. Substituting such a $\zeta(x, s)$ and $\rho(x, s)$ in (3.2.7) and (3.2.8), respectively, we obtain

$$
\begin{equation*}
\int_{0}^{n} v(x, t) \chi_{1}(x) d x \leq \tau_{1} \int_{0}^{t} \int_{0}^{\infty} v(x, s)^{+} d x d s+\tau_{2} \int_{0}^{t} \int_{0}^{\infty} w(x, s)^{+} d x d s \tag{3.2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{n} w(x, t) \chi_{2}(x) d x \leq \tau_{3} \int_{0}^{t} \int_{0}^{\infty} w(x, s)^{+} d x d s+\tau_{4} \int_{0}^{t} \int_{0}^{\infty} v(x, s)^{+} d x d s \tag{3.2.10}
\end{equation*}
$$

where

$$
\begin{aligned}
& \tau_{1}= \sup \left\{\gamma_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right)+\left[\lambda_{1}-\int_{0}^{\infty} \beta(x, y) \bar{u}(y, s) d y-m_{1}\left(x, s, \varphi^{\bar{u}}, \varphi^{\bar{z}}\right)\right]\right. \\
&\left.+\frac{1}{2} \int_{0}^{\infty} \beta(z, x) \underline{u}(z, s) d z+\left(\frac{3}{2}\|\beta\|_{\infty}+\left\|m_{1 \varphi^{u}}\right\|_{\infty}+\left\|\gamma_{1 \varphi^{u}}\right\|_{\infty}\right) \int_{0}^{\infty} \bar{u}(y, s) d y\right\} \\
& \tau_{2}=\left(\left\|\gamma_{1 \varphi^{z}}\right\|_{\infty}+\left\|m_{1 \varphi^{z}}\right\|_{\infty}\right) \sup \left\{\int_{0}^{\infty} \bar{u}(x, s) d x\right\}, \\
& \tau_{3}= \sup \left\{\gamma_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right)+\left[\lambda_{2}-m_{2}\left(x, s, \varphi^{\underline{u}}, \varphi^{\bar{z}}\right)\right]+\left(\left\|\gamma_{2 \varphi^{z}}\right\|_{\infty}+\left\|m_{2 \varphi^{z}}\right\|_{\infty}\right) \int_{0}^{\infty} \bar{z}(y, s) d y\right\},
\end{aligned}
$$

and

$$
\tau_{4}=\left(\left\|\gamma_{2 \varphi^{u}}\right\|_{\infty}+\left\|m_{2 \varphi^{u}}\right\|_{\infty}\right) \sup \left\{\int_{0}^{\infty} \bar{z}(x, s) d x\right\}
$$

Since (3.2.9) holds for every $\chi_{1}$, we can choose a sequence $\left\{\chi_{k}^{1}\right\}$ on $(0, n)$ converging to

$$
\chi_{1}= \begin{cases}1, & \text { if } w(x, t)>0 \\ 0, & \text { otherwise }\end{cases}
$$

Hence, by (3.2.9) we have

$$
\begin{equation*}
\int_{0}^{n} v(x, t)^{+} d x \leq \tau_{1} \int_{0}^{t} \int_{0}^{\infty} v(x, s)^{+} d x d s+\tau_{2} \int_{0}^{t} \int_{0}^{\infty} w(x, s)^{+} d x d s \tag{3.2.11}
\end{equation*}
$$

In the same fashion, by (3.2.10) we have

$$
\begin{equation*}
\int_{0}^{n} w(x, t)^{+} d x \leq \tau_{3} \int_{0}^{t} \int_{0}^{\infty} w(x, s)^{+} d x d s+\tau_{4} \int_{0}^{t} \int_{0}^{\infty} v(x, s)^{+} d x d s \tag{3.2.12}
\end{equation*}
$$

Note that $\tau_{1}, \tau_{2}, \tau_{3}$ and $\tau_{4}$ are independent of $n$, by letting $n \rightarrow \infty$ in (3.2.11) and (3.2.12), respectively, we obtain

$$
\int_{0}^{\infty} v(x, t)^{+} d x \leq \tau_{1} \int_{0}^{t} \int_{0}^{\infty} v(x, s)^{+} d x d s+\tau_{2} \int_{0}^{t} \int_{0}^{\infty} w(x, s)^{+} d x d s
$$

and

$$
\int_{0}^{\infty} w(x, t)^{+} d x \leq \tau_{3} \int_{0}^{t} \int_{0}^{\infty} w(x, s)^{+} d x d s+\tau_{4} \int_{0}^{t} \int_{0}^{\infty} v(x, s)^{+} d x d s
$$

Let $\tau=\max \left\{\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right\}$, then we get

$$
\int_{0}^{\infty}\left(v(x, t)^{+}+w(x, t)^{+}\right) d x \leq 2 \tau \int_{0}^{t} \int_{0}^{\infty}\left(v(x, s)^{+}+w(x, s)^{+}\right) d x d s
$$

By the Gronwall's inequality, we obtain

$$
\int_{0}^{\infty}\left(v(x, t)^{+}+w(x, t)^{+}\right) d x=0
$$

which implies the estimates.

Remark 3.2.1. From the proof of Theorem 3.2.1, it easily follows that for any function $\phi \in \mathbf{L}^{\infty}\left((0, T) ; \mathbf{L}^{1}(0, \infty)\right)$, if $\phi(x, 0) \leq 0$ a.e. in $(0, \infty)$, and the following inequality holds for every non-negative $\xi \in \mathbf{C}_{0, r}^{1}\left(\mathbb{D}_{T}\right)$

$$
\begin{align*}
& \int_{0}^{\infty} \quad \phi(x, t) \xi(x, t) d x \leq \int_{0}^{\infty} \phi(x, 0) \xi(x, 0) d x+\int_{0}^{t} \xi(0, s) \int_{0}^{\infty} A(x, s) \phi(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g(x, s) \xi_{x}(x, s)\right] \phi(x, s) d x d s+\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) B(x, s) \phi(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} \phi(x, s) \int_{0}^{\infty} \xi(x+y, s) C(x, y, s) d y d x d s \tag{3.2.13}
\end{align*}
$$

with $A, B \in \mathbf{L}^{\infty}\left(\mathbb{D}_{T}\right), \int_{0}^{\infty} C(x, y, t) d y \in \mathbf{L}^{\infty}\left(\mathbb{D}_{T}\right)$, and $A, C \geq 0$. Then we have $\phi(x, t) \leq 0$ a.e. in $\mathbb{D}_{T}$.

### 3.3 Existence Of the Solution

We begin this section by constructing monotone sequences of upper and lower solutions. Suppose that $\left(\bar{u}^{0}, \bar{z}^{0}\right)$ and $\left(\underline{u}^{0}, \underline{z}^{0}\right)$ are a pair of upper and lower solutions of (3.1.1), then by Theorem 3.2.1 we see that $\underline{u}^{0}(x, t) \leq \bar{u}^{0}(x, t)$ and $\underline{z}^{0}(x, t) \leq \bar{z}^{0}(x, t)$. We then set up four sequences $\left\{\underline{u}^{k}\right\}_{k=0}^{\infty},\left\{\bar{u}^{k}\right\}_{k=0}^{\infty},\left\{\underline{z}^{k}\right\}_{k=0}^{\infty}$ and $\left\{\bar{z}^{k}\right\}_{k=0}^{\infty}$ by the following procedure:

For $k=1,2, \ldots$, let $\underline{u}^{k}$ and $\bar{u}^{k}$ satisfy the system

$$
\begin{align*}
& \underline{u}_{t}^{k}+\left(g_{1} \underline{u}^{k}\right)_{x}+m_{1}\left(x, t, \varphi^{\bar{u}^{k-1}}, \varphi^{\bar{z}^{k-1}}\right) \underline{u}^{k} \\
& \quad=\frac{1}{2} \int_{0}^{x} \beta(x-y, y) \underline{u}^{k-1}(x-y, t) \underline{u}^{k-1}(y, t) d y-\underline{u}^{k}(x, t) \int_{0}^{\infty} \beta(x, y) \bar{u}^{k-1}(y, t) d y \\
& g_{1}(0, t) \underline{u}^{k}(0, t)=\int_{0}^{\infty} \gamma_{1}\left(y, t, \varphi^{\bar{u}^{k-1}}, \varphi^{\bar{z}^{k-1}}\right) \underline{u}^{k-1}(y, t) d y \\
& \underline{u}^{k}(x, 0)=u_{0}(x), \tag{3.3.1}
\end{align*}
$$

and

$$
\begin{align*}
& \bar{u}_{t}^{k}+\left(g_{1} \bar{u}^{k}\right)_{x}+m_{1}\left(x, t, \varphi^{\underline{u}^{k-1}}, \varphi^{\underline{z}^{k-1}}\right) \bar{u}^{k} \\
& \quad=\frac{1}{2} \int_{0}^{x} \beta(x-y, y) \bar{u}^{k-1}(x-y, t) \bar{u}^{k-1}(y, t) d y-\bar{u}^{k}(x, t) \int_{0}^{\infty} \beta(x, y) \underline{u}^{k-1}(y, t) d y \\
& g_{1}(0, t) \bar{u}^{k}(0, t)=\int_{0}^{\infty} \gamma_{1}\left(y, t, \varphi^{\underline{u}^{k-1}}, \varphi^{z^{k-1}}\right) \bar{u}^{k-1}(y, t) d y \\
& \bar{u}^{k}(x, 0)=u_{0}(x) . \tag{3.3.2}
\end{align*}
$$

For $k=1,2, \ldots$, let $\underline{z}^{k}$ and $\bar{z}^{k}$ satisfy the system

$$
\begin{align*}
& \underline{z}_{t}^{k}+\left(g_{2} \underline{z}^{k}\right)_{x}+m_{2}\left(x, t, \varphi^{u^{k-1}}, \varphi^{\bar{z}^{k-1}}\right) \underline{z}^{k}=0 \\
& g_{2}(0, t) \underline{z}^{k}(0, t)=\int_{0}^{\infty} \gamma_{2}\left(y, t, \varphi^{\underline{u}^{k-1}}, \varphi^{\bar{z}^{k-1}}\right) \underline{k}^{k-1}(y, t) d y  \tag{3.3.3}\\
& \underline{z}^{k}(x, 0)=z_{0}(x),
\end{align*}
$$

and

$$
\begin{align*}
& \bar{z}_{t}^{k}+\left(g_{2} \bar{z}^{k}\right)_{x}+m_{2}\left(x, t, \varphi^{\bar{u}^{k-1}}, \varphi^{z^{k-1}}\right) \bar{z}^{k}=0 \\
& g_{2}(0, t) \bar{z}^{k}(0, t)=\int_{0}^{\infty} \gamma_{2}\left(y, t, \varphi^{\bar{u}^{k-1}}, \varphi^{z^{k-1}}\right) \bar{z}^{k-1}(y, t) d y  \tag{3.3.4}\\
& \bar{z}^{k}(x, 0)=z_{0}(x) .
\end{align*}
$$

The existence of solution to problems (3.3.1)-(3.3.4) follows from the fact that (3.3.1)(3.3.4) are all linear problems with local boundary conditions.

We first show that

$$
\begin{equation*}
\underline{u}^{0} \leq \underline{u}^{1}, \bar{u}^{1} \leq \bar{u}^{0}, \underline{z}^{0} \leq \underline{z}^{1}, \text { and } \bar{z}^{1} \leq \bar{z}^{0} \text { a.e in } \mathbb{D}_{T} . \tag{3.3.5}
\end{equation*}
$$

Let $v(x, t)=\underline{u}^{0}(x, t)-\underline{u}^{1}(x, t)$ and $w(x, t)=\underline{z}^{0}(x, t)-\underline{z}^{1}(x, t)$, then by (3.3.1), (3.3.3) and the fact that $\left(\underline{u}^{0}(x, t), \underline{z}^{0}(x, t)\right)$ is the lower solution of (3.1.1), we obtain

$$
\begin{aligned}
& \int_{0}^{\infty} v(x, t) \xi(x, t) d x \\
\leq & \int_{0}^{\infty} v(x, 0) \xi(x, 0) d x+\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g_{1}(x, s) \xi_{x}(x, s)\right] v(x, s) d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty} v(x, s) \xi(x, s)\left[\int_{0}^{\infty} \beta(x, y) \bar{u}^{0}(y, s) d y+m_{1}\left(x, s, \varphi^{\bar{u}^{0}}, \varphi^{\bar{z}^{0}}\right)\right] d x d s
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{0}^{\infty} w(x, t) \eta(x, t) d x \\
\leq & \int_{0}^{\infty} w(x, 0) \eta(x, 0) d x+\int_{0}^{t} \int_{0}^{\infty}\left[\eta_{s}(x, s)+g_{2}(x, s) \eta_{x}(x, s)\right] w(x, s) d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty} w(x, s) \eta(x, s) m_{2}\left(x, s, \varphi^{u^{0}}, \varphi^{\bar{z}^{0}}\right) d x d s
\end{aligned}
$$

Then $v(x, t)$ satisfies (3.2.13) with $A(x, t)=0, B(x, t)=-\int_{0}^{\infty} \beta(x, y) \bar{u}^{0}(y, s) d y-$ $m_{1}\left(x, s, \varphi^{\bar{u}^{0}}, \varphi^{\bar{z}^{0}}\right)$ and $C(x, y, t)=0$, and $w(x, t)$ satisfies (3.2.13) with $A(x, t)=0$, $B(x, t)=-m_{2}\left(x, s, \varphi^{u^{0}}, \varphi^{z^{0}}\right)$ and $C(x, y, t)=0$. Thus, by Remark 3.2.1 we obtain that $v(x, t) \leq 0$ and $w(x, t) \leq 0$ a.e in $\mathbb{D}_{T}$, i.e., $\underline{u}^{0} \leq \underline{u}^{1}$ and $\underline{z}^{0} \leq \underline{z}^{1}$ a.e in $\mathbb{D}_{T}$. In a similar manner, we can show that $\bar{u}^{1} \leq \bar{u}^{0}$ and $\bar{z}^{1} \leq \bar{z}^{0}$ a.e in $\mathbb{D}_{T}$.

We then show that $\left(\underline{u}^{1}, \underline{z}^{1}\right)$ and $\left(\bar{u}^{1}, \bar{z}^{1}\right)$ are the lower and upper solutions of (3.1.1), respectively. By (3.3.5), $m_{1 \varphi^{u}} \geq 0$ and $m_{1 \varphi^{z}} \geq 0$, we find

$$
\begin{aligned}
& \frac{1}{2} \int_{0}^{x} \beta(x-y, y) \underline{u}^{0}(x-y, t) \underline{u}^{0}(y, t) d y-\underline{u}^{1}(x, t) \int_{0}^{\infty} \beta(x, y) \bar{u}^{0}(y, t) d y-m_{1}\left(x, t, \varphi^{\bar{u}^{0}}, \varphi^{\bar{z}^{0}}\right) \underline{u}^{1} \\
\leq & \frac{1}{2} \int_{0}^{x} \beta(x-y, y) \underline{u}^{1}(x-y, t) \underline{u}^{1}(y, t) d y-\underline{u}^{1}(x, t) \int_{0}^{\infty} \beta(x, y) \bar{u}^{1}(y, t) d y-m_{1}\left(x, t, \varphi^{\bar{u}^{1}}, \varphi^{\bar{z}^{1}}\right) \underline{u}^{1},
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{2} \int_{0}^{x} \beta(x-y, y) \bar{u}^{0}(x-y, t) \bar{u}^{0}(y, t) d y-\bar{u}^{1}(x, t) \int_{0}^{\infty} \beta(x, y) \underline{u}^{0}(y, t) d y-m_{1}\left(x, t, \varphi^{\underline{u}^{0}}, \varphi^{z^{0}}\right) \bar{u}^{1} \\
\geq & \frac{1}{2} \int_{0}^{x} \beta(x-y, y) \bar{u}^{1}(x-y, t) \bar{u}^{1}(y, t) d y-\bar{u}^{1}(x, t) \int_{0}^{\infty} \beta(x, y) \underline{u}^{1}(y, t) d y-m_{1}\left(x, t, \varphi^{\underline{u}^{1}}, \varphi^{z^{1}}\right) \bar{u}^{1} .
\end{aligned}
$$

By (3.3.5), $\gamma_{1 \varphi^{u}} \leq 0$ and $\gamma_{1 \varphi^{z}} \leq 0$, we find

$$
\int_{0}^{\infty} \gamma_{1}\left(y, t, \varphi^{\bar{u}^{0}}, \varphi^{\bar{z}^{0}}\right) \underline{u}^{0}(y, t) d y \leq \int_{0}^{\infty} \gamma_{1}\left(y, t, \varphi^{\bar{u}^{1}}, \varphi^{\bar{z}^{1}}\right) \underline{u}^{1}(y, t) d y
$$

and

$$
\int_{0}^{\infty} \gamma_{1}\left(y, t, \varphi^{\underline{u}^{0}}, \varphi^{\underline{z}^{0}}\right) \bar{u}^{0}(y, t) d y \geq \int_{0}^{\infty} \gamma_{1}\left(y, t, \varphi^{\underline{u}^{1}}, \varphi^{z^{1}}\right) \bar{u}^{1}(y, t) d y
$$

By (3.3.5), $m_{2 \varphi^{u}} \leq 0$ and $m_{2 \varphi^{z}} \geq 0$, we obtain

$$
-m_{2}\left(x, t, \varphi^{\underline{u}^{0}}, \varphi^{\bar{z}^{0}}\right) \underline{z}^{1} \leq-m_{2}\left(x, t, \varphi^{\underline{u}^{1}}, \varphi^{\bar{z}^{1}}\right) \underline{z}^{1},
$$

and

$$
-m_{2}\left(x, t, \varphi^{\bar{u}^{0}}, \varphi^{z^{0}}\right) \bar{z}^{1} \geq-m_{2}\left(x, t, \varphi^{\bar{u}^{1}}, \varphi^{z^{1}}\right) \bar{z}^{1}
$$

By (3.3.5), $\gamma_{2 \varphi^{u}} \geq 0$ and $\gamma_{2 \varphi^{z}} \leq 0$, we obtain

$$
\int_{0}^{\infty} \gamma_{2}\left(y, t, \varphi^{u^{0}}, \varphi^{\bar{z}^{0}}\right) \underline{z}^{0}(y, t) d y \leq \int_{0}^{\infty} \gamma_{2}\left(y, t, \varphi^{\underline{u}^{1}}, \varphi^{\bar{z}^{1}}\right) \underline{z}^{1}(y, t) d y,
$$

and

$$
\int_{0}^{\infty} \gamma_{2}\left(y, t, \varphi^{\bar{u}^{0}}, \varphi^{\underline{z}^{0}}\right) \bar{z}^{0}(y, t) d y \geq \int_{0}^{\infty} \gamma_{2}\left(y, t, \varphi^{\bar{u}^{1}}, \varphi^{z^{1}}\right) \bar{z}^{1}(y, t) d y .
$$

Thus, $\left(\underline{u}^{1}, \underline{z}^{1}\right)$ and $\left(\bar{u}^{1}, \bar{z}^{1}\right)$ are the lower and upper solutions of (3.1.1), respectively.
We then assumed that for some $k>1,\left(\underline{u}^{k}, \underline{z}^{k}\right)$ and $\left(\bar{u}^{k}, \bar{z}^{k}\right)$ are the lower and upper solutions of (3.1.1), respectively. Proceeding analogously, we can show that

$$
\underline{u}^{k} \leq \underline{u}^{k+1}, \bar{u}^{k+1} \leq \bar{u}^{k}, \underline{z}^{k} \leq \underline{z}^{k+1}, \text { and } \bar{z}^{k+1} \leq \bar{z}^{k} \text { a.e in } \mathbb{D}_{T} .
$$

and then by the above inequalities we can claim that $\left(\underline{u}^{k+1}, \underline{z}^{k+1}\right)$ and $\left(\bar{u}^{k+1}, \bar{z}^{k+1}\right)$ are the lower and upper solutions of (3.1.1), respectively. Thus, we obtain four monotone sequences

$$
\begin{aligned}
& \underline{u}^{0} \leq \underline{u}^{1} \leq \cdots \leq \underline{u}^{k} \leq \bar{u}^{k} \leq \cdots \leq \bar{u}^{1} \leq \bar{u}^{0}, \text { a.e. in } \mathbb{D}_{T}, \\
& \underline{z}^{0} \leq \underline{z}^{1} \leq \cdots \leq \underline{z}^{k} \leq \bar{z}^{k} \leq \cdots \leq \bar{z}^{1} \leq \bar{z}^{0}, \text { a.e. in } \mathbb{D}_{T}
\end{aligned}
$$

for $k=0,1,2, \ldots$ By the monotonicity of the sequences $\left\{\underline{u}^{k}\right\}_{k=0}^{\infty},\left\{\bar{u}^{k}\right\}_{k=0}^{\infty},\left\{\underline{z}^{k}\right\}_{k=0}^{\infty}$ and $\left\{\bar{z}^{k}\right\}_{k=0}^{\infty}$, we know that there exist functions $\underline{u}, \bar{u}, \underline{z}$ and $\bar{z}$ such that $\underline{u}^{k} \rightarrow \underline{u}, \bar{u}^{k} \rightarrow \bar{u}$, $\underline{z}^{k} \rightarrow \underline{z}, \bar{z}^{k} \rightarrow \bar{z}$ pointwise in $\mathbb{D}_{T}$. Clearly, $\underline{u} \leq \bar{u}$ and $\underline{z} \leq \bar{u}$ a.e. in $\mathbb{D}_{T}$. By the dominant
convergence theorem, we know that $(\bar{u}, \bar{z})$ and $(\underline{u}, \underline{z})$ are the lower solution and upper solution of (3.1.1), respectively. Hence by Theorem 3.2.1, we have $\bar{u} \leq \underline{u}$ and $\bar{z} \leq \underline{z}$ a.e. in $\mathbb{D}_{T}$. Thus, $\bar{u}=\underline{u}, \bar{z}=\underline{z}$ a.e. in $\mathbb{D}_{T}$. Let $u=\bar{u}$ and $z=\bar{z}$, then $(u(x, t), z(x, t))$ is the solution of (3.1.1).

Remark 3.3.1. As an example, for a large class of initial data such as $u_{0}(x)=O\left(e^{-x}\right)$ and $z_{0}(x)=O\left(e^{-x}\right)$ as $x \rightarrow \infty$, we can construct a pair of non-negative lower and upper solutions of (3.1.1) as follows: let $\left(\underline{u}^{0}(x, t), \underline{z}^{0}(x, t)\right)=(0,0)$ and $\left(\bar{u}^{0}(x, t), \bar{z}^{0}(x, t)\right)=$ $\left(\frac{c_{1} e^{b_{1} t}}{1+a_{1}^{2} x^{2}}, \frac{c_{2} e^{b_{2} t}}{1+a_{2}^{2} x^{2}}\right)$ with $a_{i}, b_{i}, c_{i}$ positive constants, $i=1,2$. For every non-negative $\xi \in$ $\mathbf{C}_{0, r}^{1}\left(\mathbb{D}_{T}\right)$, we find

$$
\begin{align*}
& \int_{0}^{\infty} \bar{u}^{0}(x, 0) \xi(x, 0) d x+\int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{\underline{u}^{0}}, \varphi^{\underline{z}^{0}}\right) \bar{u}^{0}(x, s) d x d s \\
&+\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g_{1}(x, s) \xi_{x}(x, s)\right] \bar{u}^{0}(x, s) d x d s \\
&+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y) \bar{u}^{0}(x-y, s) \bar{u}^{0}(y, s) d y d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} \xi(x, s)\left[\int_{0}^{\infty} \beta(x, y) \underline{u}^{0}(y, s) d y+m_{1}\left(x, s, \varphi^{\underline{u}^{0}}, \varphi^{z^{0}}\right)\right] \bar{u}^{0}(x, s) d x d s \\
&= \int_{0}^{\infty} \bar{u}^{0}(x, 0) \xi(x, 0) d x+\int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{1}(x, s, 0,0) \bar{u}^{0}(x, s) d x d s \\
&-\int_{0}^{t} g_{1}(0, s) \bar{u}^{0}(0, s) \xi(0, s) d s-\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \bar{u}^{0}(x, s)\left(g_{x}^{1}(x, s)-g_{1}(x, s) \frac{2 a_{1}^{2} x}{1+a_{1}^{2} x^{2}}\right) d x d s \\
&+\int_{0}^{\infty}\left(\bar{u}^{0}(x, t) \xi(x, t)-\bar{u}^{0}(x, 0) \xi(x, 0)-\int_{0}^{t} b_{1} \xi(x, s) \bar{u}^{0}(x, s) d s\right) d x \\
&+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y) \bar{u}^{0}(x-y, s) \bar{u}^{0}(y, s) d y d x d s \\
&-\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) m_{1}(x, s, 0,0) \bar{u}^{0}(x, s) d x d s \\
& \leq \int_{0}^{\infty} \bar{u}^{0}(x, t) \xi(x, t) d x+\int_{0}^{t} \xi(0, s)\left[-g_{1}(0, s) \bar{u}^{0}(0, s)+\int_{0}^{\infty} \gamma_{1}(x, s, 0,0) \bar{u}^{0}(x, s) d x\right] d s \\
&+\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \bar{u}^{0}(x, s)\left[a_{1} g_{1}(x, s)-g_{x}^{1}(x, s)-b_{1}-m_{1}(x, s, 0,0)\right] d x d s \\
&+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y) \bar{u}^{0}(x-y, s) \bar{u}^{0}(y, s) d y d x d s . \tag{3.3.6}
\end{align*}
$$

First we choose $a_{1}$ so large that $a_{1} \geq \frac{\pi\left\|\gamma_{1}\right\|_{\infty}}{2 \min _{s \in[0,1]} g_{1}(0, s)}$. Note that $\int_{0}^{\infty} \frac{1}{1+a_{1}^{2} x^{2}} d x=\frac{\pi}{2 a_{1}}$, we
obtain

$$
\begin{align*}
& -g_{1}(0, s) \bar{u}^{0}(0, s)+\int_{0}^{\infty} \gamma_{1}(x, s, 0,0) \bar{u}^{0}(x, s) d x \\
\leq & c_{1} e^{b s}\left\|\gamma_{1}\right\|_{\infty} \int_{0}^{\infty} \frac{1}{1+a_{1}^{2} x^{2}} d x-c_{1} e^{b s} g_{1}(0, s)=c_{1} e^{b s}\left(\frac{\pi\left\|\gamma_{1}\right\|_{\infty}}{2 a_{1}}-g_{1}(0, s)\right) \leq 0 . \tag{3.3.7}
\end{align*}
$$

Fix this $a_{1}$ and choose $c_{1}$ so large that $\frac{c_{1}}{1+a_{1} x^{2}} \geq u_{0}(x)$. Note that

$$
\int_{0}^{x} \frac{1}{\left[1+a_{1}^{2}(x-y)^{2}\right]\left(1+a_{1}^{2} y^{2}\right)} d y=\frac{2}{a_{1}^{2} x}\left[\frac{a_{1} x \tan ^{-1}\left(a_{1} x\right)+\log \left(1+a_{1}^{2} x^{2}\right)}{4+a_{1}^{2} x^{2}}\right] \leq \frac{2(1+\pi)}{a_{1}\left(1+a_{1}^{2} x^{2}\right)}
$$

we have

$$
\begin{aligned}
& \frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y) \bar{u}^{0}(x-y, s) \bar{u}^{0}(y, s) d y d x d s \\
\leq & \frac{3 c_{1}}{a_{1}}(1+\pi)\|\beta\|_{\infty} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \bar{u}^{0}(x, s) d x d s .
\end{aligned}
$$

We then choose $b_{1}$ sufficiently large that $b_{1} \geq \frac{3 c_{1}}{a_{1}}(1+\pi)\|\beta\|_{\infty}+a_{1}\left\|g_{1}\right\|_{\infty}+\left\|g_{1 x}\right\|_{\infty}$. Hence, we obtain

$$
\begin{align*}
& \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \bar{u}^{0}(x, s)\left[a_{1} g_{1}(x, s)-g_{1 x}(x, s)-b_{1}-m_{1}(x, s, 0,0)\right] d x d s \\
& +\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y) \bar{u}^{0}(x-y, s) \bar{u}^{0}(y, s) d y d x d s \\
\leq & \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \bar{u}^{0}(x, s)\left[a_{1} g_{1}(x, s)-g_{1 x}(x, s)-b_{1}-m_{1}(x, s, 0,0)+\frac{3 c_{1}}{a_{1}}(1+\pi)\|\beta\|_{\infty}\right] d x d s \\
\leq & 0 . \tag{3.3.8}
\end{align*}
$$

Thus, by (3.3.6)-(3.3.8), we see that $\left(\underline{u}^{0}(x, t), \underline{z}^{0}(x, t)\right)$ and $\left(\bar{u}^{0}(x, t), \bar{z}^{0}(x, t)\right)$ satisfy (3.2.1).

In the same fashion, we have

$$
\begin{align*}
& \quad \int_{0}^{\infty} \bar{z}^{0}(x, 0) \xi(x, 0) d x+\int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{\bar{u}^{0}}, \varphi^{z^{0}}\right) \bar{z}^{0}(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g_{2}(x, s) \xi_{x}(x, s)\right] \bar{z}^{0}(x, s) d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) m_{2}\left(x, s, \varphi^{\bar{u}^{0}}, \varphi^{z^{0}}\right) \bar{z}^{0}(x, s) d x d s \\
& \leq \int_{0}^{\infty} \bar{z}^{0}(x, t) \xi(x, t) d x+\int_{0}^{t} \xi(0, s)\left[-g_{2}(0, s) \bar{z}^{0}(0, s)+\int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{\bar{u}^{0}}, 0\right) \bar{z}^{0}(x, s) d x\right] d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \bar{z}^{0}(x, s)\left[a_{2} g_{2}(x, s)-g_{2 x}(x, s)-b_{2}\right] d x d s . \tag{3.3.9}
\end{align*}
$$

First we choose $a_{2}$ so large that $a_{2} \geq \frac{\pi\left\|\gamma_{2}\right\|_{\infty}}{2 \min _{s \in[0,1]} g_{2}(0, s)}$. Then we have

$$
\begin{equation*}
-g_{2}(0, s) \bar{z}^{0}(0, s)+\int_{0}^{\infty} \gamma_{2}\left(\bar{u}^{0}, 0, x, s\right) \bar{z}^{0}(x, s) d x \leq 0 \tag{3.3.10}
\end{equation*}
$$

Fix this $a_{2}$ and choose $c_{2}$ so large that $\frac{c_{2}}{1+a_{2} x^{2}} \geq z_{0}(x)$. We then choose $b_{2}$ sufficiently large that $b_{2} \geq a_{2}\left\|g_{2}\right\|_{\infty}+\left\|g_{2 x}\right\|_{\infty}$. Then we have

$$
\begin{equation*}
\int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \bar{z}^{0}(x, s)\left[a_{2} g_{2}(x, s)-g_{2 x}(x, s)-b_{2}\right] d x d s \leq 0 \tag{3.3.11}
\end{equation*}
$$

Thus, by (3.3.9)-(3.3.11), we see that $\left(\underline{u}^{0}(x, t), \underline{z}^{0}(x, t)\right)$ and $\left(\bar{u}^{0}(x, t), \bar{z}^{0}(x, t)\right)$ satisfy (3.2.3).

Therefore, $\left(\underline{u}^{0}(x, t), \underline{z}^{0}(x, t)\right)$ and $\left(\bar{u}^{0}(x, t), \bar{z}^{0}(x, t)\right)$ is a pair of lower and upper solution of (3.1.1) on $\mathbb{D}_{T}$ with $T=\min \{1,1 / b\}$.

### 3.4 Uniqueness Of the Solution

In order to show the uniqueness, we give a new definition of the solution of (3.1.1).

Definition 3.4.1. A pair of non-negative functions $(u(x, t), z(x, t))$ is called a solution of (3.1.1) on $\mathbb{D}_{T}$ if all the following hold:
(i) $u, z \in \mathbf{L}^{\infty}\left((0, T) ; \mathbf{L}^{1}(0, \infty)\right)$.
(ii) For every $t \in(0, T)$ and every $\xi, \eta \in \mathbf{C}_{0, r}^{1}\left(\mathbb{D}_{T}\right)$, we have

$$
\begin{align*}
& \int_{0}^{\infty} u(x, t) \xi(x, t) d x \\
= & \int_{0}^{\infty} u(x, 0) \xi(x, 0) d x+\int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right) u(x, s) d x d s \\
& +\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g_{1}(x, s) \xi_{x}(x, s)\right] u(x, s) d x d s  \tag{3.4.1}\\
& +\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \xi(x, s) \int_{0}^{x} \beta(x-y, y) u(x-y, s) u(y, s) d y d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty} \xi(x, s)\left[\int_{0}^{\infty} \beta(x, y) u(y, s) d y+m_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right)\right] u(x, s) d x d s,
\end{align*}
$$

and

$$
\begin{align*}
& \int_{0}^{\infty} z(x, t) \eta(x, t) d x \\
= & \int_{0}^{\infty} z(x, 0) \eta(x, 0) d x+\int_{0}^{t} \eta(0, s) \int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{u}, \varphi^{z}\right) z(x, s) d x d s  \tag{3.4.2}\\
& +\int_{0}^{t} \int_{0}^{\infty}\left[\eta_{s}(x, s)+g_{2}(x, s) \eta_{x}(x, s)\right] z(x, s) d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty} m_{2}\left(x, s, \varphi^{u}, \varphi^{z}\right) z(x, s) \eta(x, s) d x d s
\end{align*}
$$

We can easily verify that this definition is equivalent to Definition 3.2.2.
Theorem 3.4.1. Suppose that $\left(u^{1}, z^{1}\right)$ and $\left(u^{2}, z^{2}\right)$ are both the solutions of (3.1.1), then $u^{1} \equiv u^{2}$ and $z^{1} \equiv z^{2}$.

Proof. Let $v=u^{1}-u^{2}$ and $w=z^{1}-z^{2}$. Choose $\xi$ and $\eta \in \mathbf{C}_{0, r}^{1}((0, n) \times(0, T))$. Then $v$ and $w$ satisfy $v(x, 0)=0$ and $w(x, 0)=0$ in $[0, \infty)$. By (3.2.1) and (3.2.2), we find

$$
\begin{align*}
& \int_{0}^{\infty} v(x, t) \xi(x, t) d x \\
= & \int_{0}^{t} \xi(0, s) \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right) v(x, s) d x d s+\int_{0}^{t} \int_{0}^{\infty}\left[\xi_{s}(x, s)+g_{1}(x, s) \xi_{x}(x, s)\right] v(x, s) d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty} \xi(x, s)\left[\int_{0}^{\infty} \beta(x, y) u^{1}(y, s) d y+m_{1}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right)\right] v(x, s) d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty} u^{2}(x, s) \int_{0}^{\infty} \xi(x, s) \beta(x, y) v(y, s) d y d x d s \\
& +\int_{0}^{t} \int_{0}^{\infty} u^{2}(x, s)\left[\xi(0, s) \gamma_{1 \varphi^{u}}\left(x, s, \theta_{u_{\gamma}}, \varphi^{z^{1}}\right)-\xi(x, s) m_{1 \varphi^{u}}\left(x, s, \theta_{u_{m}}, \varphi^{z^{1}}\right)\right] \int_{0}^{\infty} v(y, s) d y d x d s \\
& +\int_{0}^{t} \int_{0}^{\infty} u^{2}(x, s)\left[\xi(0, s) \gamma_{1 \varphi^{z}}\left(x, s, \varphi^{u^{2}}, \theta_{z_{\gamma}}\right)-\xi(x, s) m_{1 \varphi^{z}}\left(x, s, \varphi^{u^{2}}, \theta_{z_{m}}\right)\right] \int_{0}^{\infty} w(y, s) d y d x d s \\
& +\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} v(y, s) \int_{0}^{\infty} \xi(y+z, s) \beta(z, y) u^{1}(z, s) d z d y d s \\
& +\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} u^{2}(y, s) \int_{0}^{\infty} \xi(y+z, s) \beta(z, y) v(z, s) d z d y d s, \tag{3.4.3}
\end{align*}
$$

and

$$
\begin{align*}
& \int_{0}^{\infty} w(x, t) \eta(x, t) d x=\int_{0}^{t} \eta(0, s) \int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right) w(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\eta(0, s) \gamma_{2 \varphi^{u}}\left(x, s, \theta_{1}, \varphi^{z^{1}}\right)-\eta(x, s) m_{2 \varphi^{u}}\left(x, s, \theta_{2}, \varphi^{z^{1}}\right)\right] z^{2}(x, s) \int_{0}^{\infty} v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\eta(0, s) \gamma_{2 \varphi^{z}}\left(x, s, \varphi^{u^{2}}, \theta_{3}\right)-\eta(x, s) m_{2 \varphi^{z}}\left(x, s, \varphi^{u^{2}}, \theta_{4}\right)\right] z^{2}(x, s) \int_{0}^{\infty} w(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\eta_{s}(x, s)+g_{2}(x, s) \eta_{x}(x, s)-\eta(x, s) m_{2}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right)\right] w(x, s) d x d s, \tag{3.4.4}
\end{align*}
$$

where $\theta_{1}$ and $\theta_{2}$ are both between $\varphi^{u^{1}}$ and $\varphi^{u^{2}}$, and $\theta_{3}$ and $\theta_{4}$ are both between $\varphi^{z^{1}}$ and $\varphi^{z^{2}}$. Let $\xi(x, t)=e^{\lambda_{1} t} \zeta(x, t)$, where $\zeta \in \mathbf{C}_{0, r}^{1}((0, n) \times(0, T))$. Then by (3.4.3), we obtain

$$
\begin{align*}
& e^{\lambda_{1} t} \int_{0}^{\infty} v(x, t) \zeta(x, t) d x=\int_{0}^{t} e^{\lambda_{1} s} \zeta(0, s) \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right) v(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\zeta_{s}(x, s)+g_{1}(x, s) \zeta_{x}(x, s)\right] e^{\lambda_{1} s} v(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} \zeta(x, s) v(x, s)\left[\lambda_{1}-\int_{0}^{\infty} \beta(x, y) u^{1}(y, s) d y-m_{1}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right)\right] d x d s \\
& \quad-\int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} u^{2}(x, s) \int_{0}^{\infty} \zeta(x, s) \beta(x, y) v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\zeta(0, s) \gamma_{1 \varphi^{u}}\left(x, s, \theta_{u_{\gamma}}, \varphi^{z^{1}}\right)-\zeta(x, s) m_{1 \varphi^{u}}\left(x, s, \theta_{u_{m}}, \varphi^{z^{1}}\right)\right] e^{\lambda_{1} s} u^{2}(x, s) \int_{0}^{\infty} v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\zeta(0, s) \gamma_{1 \varphi^{z}}\left(x, s, \varphi^{u^{2}}, \theta_{z_{\gamma}}\right)-\zeta(x, s) m_{1 \varphi^{z}}\left(x, s, \varphi^{u^{2}}, \theta_{z_{m}}\right)\right] e^{\lambda_{1} s} u^{2}(x, s) \int_{0}^{\infty} w(y, s) d y d x d s \\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} v(y, s) \int_{0}^{\infty} \zeta(y+z, s) \beta(z, y) u^{1}(z, s) d z d y d s \\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} e^{\lambda_{1} s} u^{2}(y, s) \int_{0}^{\infty} \zeta(y+z, s) \beta(z, y) v(z, s) d z d y d s . \tag{3.4.5}
\end{align*}
$$

Let $\eta(x, t)=e^{\lambda_{2} t} \rho(x, t)$, where $\rho \in \mathbf{C}_{0, r}^{1}((0, n) \times(0, T))$. Then by (3.4.4), we find

$$
\begin{align*}
& e^{\lambda_{2} t} \int_{0}^{\infty} w(x, t) \rho(x, t) d x=\int_{0}^{t} e^{\lambda_{2} s} \rho(0, s) \int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right) w(x, s) d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\rho(0, s) \gamma_{2 \varphi^{u}}\left(x, s, \theta_{1}, \varphi^{z^{1}}\right)-\rho(x, s) m_{2 \varphi^{u}}\left(x, s, \theta_{2}, \varphi^{z^{1}}\right)\right] e^{\lambda_{2} s} z^{2}(x, s) \int_{0}^{\infty} v(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\rho(0, s) \gamma_{2 \varphi^{z}}\left(x, s, \varphi^{u^{2}}, \theta_{3}\right)-\rho(x, s) m_{2 \varphi^{z}}\left(x, s, \varphi^{u^{2}}, \theta_{4}\right)\right] e^{\lambda_{2} s} z^{2}(x, s) \int_{0}^{\infty} w(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\rho_{s}(x, s)+g_{2}(x, s) \rho_{x}(x, s)+\rho(x, s)\left(\lambda_{2}-m_{2}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right)\right)\right] e^{\lambda_{2} s} w(x, s) d x d s \tag{3.4.6}
\end{align*}
$$

We now set up two backward problems as follows:

$$
\begin{array}{ll}
\zeta_{s}(x, s)+g_{1}(x, s) \zeta_{x}(x, s)=0, & 0<s<t, 0<x<n \\
\zeta(n, s)=0, & 0<s<t \\
\zeta(x, t)=\chi_{1}(x), & 0 \leq x \leq n,
\end{array}
$$

and

$$
\begin{array}{ll}
\rho_{s}(x, s)+g_{2}(x, s) \rho_{x}(x, s)=0, & 0<s<t, 0<x<n \\
\rho(n, s)=0, & 0<s<t \\
\rho(x, t)=\chi_{2}(x), & 0 \leq x \leq n,
\end{array}
$$

where $\chi_{1}, \chi_{2} \in \mathbf{C}_{0}^{\infty}(0, n)$ and $-1 \leq \chi_{1}, \chi_{2} \leq 1$. The existence of $\zeta(x, s)$ and $\rho(x, s)$ can be easily shown. Note that the initial and boundary condition of $\zeta(x, s)$ and $\rho(x, s)$, we have $-1 \leq \zeta(x, s) \leq 1$ and $-1 \leq \rho(x, s) \leq 1$. Substituting such a $\zeta(x, s)$ and $\rho(x, s)$ in (3.4.5) and (3.4.6), respectively, we obtain

$$
\begin{equation*}
\int_{0}^{n} v(x, t) \chi_{1}(x) d x \leq \mu_{1} \int_{0}^{t} \int_{0}^{\infty}|v(x, s)| d x d s+\mu_{2} \int_{0}^{t} \int_{0}^{\infty}|w(x, s)| d x d s \tag{3.4.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{n} w(x, t) \chi_{2}(x) d x \leq \mu_{3} \int_{0}^{t} \int_{0}^{\infty}|w(x, s)| d x d s+\mu_{4} \int_{0}^{t} \int_{0}^{\infty}|v(x, s)| d x d s \tag{3.4.8}
\end{equation*}
$$

where

$$
\begin{gathered}
\mu_{1}=\sup \left\{\gamma_{1}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right)+\left[\lambda_{1}-\int_{0}^{\infty} \beta(x, y) u^{1}(y, s) d y-m_{1}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right)\right]\right. \\
\left.+\frac{1}{2} \int_{0}^{\infty} \beta(z, x) u^{1}(z, s) d z+\left(\frac{3}{2}\|\beta\|_{\infty}+\left\|m_{1 \varphi^{u}}\right\|_{\infty}+\left\|\gamma_{1 \varphi^{u}}\right\|_{\infty}\right) \int_{0}^{\infty} u^{2}(y, s) d y\right\}, \\
\mu_{2}=\left(\left\|\gamma_{1 \varphi^{z}}\right\|_{\infty}+\left\|m_{1 \varphi^{z}}\right\|_{\infty}\right) \sup \left\{\int_{0}^{\infty} u^{2}(x, s) d x\right\}, \\
\mu_{3}=\sup \left\{\gamma_{2}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right)+\left[\lambda_{2}-m_{2}\left(x, s, \varphi^{u^{1}}, \varphi^{z^{1}}\right)\right]+\left(\left\|m_{2 \varphi^{z}}\right\|_{\infty}+\left\|\gamma_{2 \varphi^{z}}\right\|_{\infty}\right) \int_{0}^{\infty} z^{2}(y, s) d y\right\},
\end{gathered}
$$

and

$$
\mu_{4}=\left(\left\|\gamma_{2 \varphi^{u}}\right\|_{\infty}+\left\|m_{2 \varphi^{u}}\right\|_{\infty}\right) \sup \left\{\int_{0}^{\infty} z^{2}(x, s) d x\right\}
$$

Since (3.4.7) holds for every $\chi_{1}$, we can choose a sequence $\left\{\chi_{k}^{1}\right\}$ on $(0, n)$ converging to

$$
\chi_{1}= \begin{cases}1, & \text { if } w(x, t)>0 \\ 0, & \text { if } w(x, t)=0 \\ -1, & \text { if } w(x, t)<0\end{cases}
$$

Hence, by (3.4.7) we have

$$
\begin{equation*}
\int_{0}^{n}|v(x, t)| d x \leq \mu_{1} \int_{0}^{t} \int_{0}^{\infty}|v(x, s)| d x d s+\mu_{2} \int_{0}^{t} \int_{0}^{\infty}|w(x, s)| d x d s \tag{3.4.9}
\end{equation*}
$$

In the same fashion, by (3.4.8) we have

$$
\begin{equation*}
\int_{0}^{n}|w(x, t)| d x \leq \mu_{3} \int_{0}^{t} \int_{0}^{\infty}|w(x, s)| d x d s+\mu_{4} \int_{0}^{t} \int_{0}^{\infty}|v(x, s)| d x d s \tag{3.4.10}
\end{equation*}
$$

Note that $\mu_{1}, \mu_{2}, \mu_{3}$ and $\mu_{4}$ are independent of $n$, by letting $n \rightarrow \infty$ in (3.4.9) and (3.4.10), respectively, we obtain

$$
\int_{0}^{\infty}|v(x, t)| d x \leq \mu_{1} \int_{0}^{t} \int_{0}^{\infty}|v(x, s)| d x d s+\mu_{2} \int_{0}^{t} \int_{0}^{\infty}|w(x, s)| d x d s
$$

and

$$
\int_{0}^{\infty}|w(x, t)| d x \leq \mu_{3} \int_{0}^{t} \int_{0}^{\infty}|w(x, s)| d x d s+\mu_{4} \int_{0}^{t} \int_{0}^{\infty}|v(x, s)| d x d s
$$

Let $\mu=\max \left\{\mu_{1}, \mu_{2}, \mu_{3}, \mu_{4}\right\}$, then we get

$$
\int_{0}^{\infty}(|v(x, t)|+|w(x, t)|) d x \leq 2 \mu \int_{0}^{t} \int_{0}^{\infty}(|v(x, s)|+|w(x, s)|) d x d s
$$

By the Gronwall's inequality, we obtain

$$
\int_{0}^{\infty}(|v(x, t)|+|w(x, t)|) d x=0
$$

which implies the estimates.

Theorem 3.4.2. Suppose that (H1)-(H5) hold. Furthermore, suppose that $\left(\underline{u}^{0}(x, t), \underline{z}^{0}(x, t)\right)$ and $\left(\bar{u}^{0}(x, t), \bar{z}^{0}(x, t)\right)$ are non-negative lower and non-negative upper solution of (3.1.1), respectively. Then there exist monotone sequences $\left\{\underline{u}^{k}(x, t), \underline{z}^{k}(x, t)\right\}$ and $\left\{\bar{u}^{k}(x, t), \bar{z}^{k}(x, t)\right\}$ which converge to the unique solution of (3.1.1).

We now show that the solution of (3.1.1) possesses the following property.

Theorem 3.4.3. Suppose that (H1)-(H5) hold. Then for the solution $(u(x, t), z(x, t))$ of (3.1.1), $\varphi^{u}(t)$ and $\varphi^{z}(t)$ are both continuous in the existence interval.

Proof. To show $\varphi^{u}(t)$ are continuous on $[0, t]$, it suffices to show that

$$
\begin{align*}
\int_{0}^{\infty} u(x, t) d x= & \int_{0}^{\infty} u(x, 0) d x+\int_{0}^{t} \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right) u(x, s) d x d s \\
& +\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \int_{0}^{x} \beta(x-y, y) u(x-y, s) u(y, s) d y d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty}\left[\int_{0}^{\infty} \beta(x, y) u(y, s) d y+m_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right)\right] u(x, s) d x d s . \tag{3.4.11}
\end{align*}
$$

Let $\xi(x, t)=\zeta(x)$, where $\zeta(x)=1$ for $0 \leq x \leq n, \zeta(x)=0$ for $n+2 \leq x<\infty$ and $-1 \leq \zeta^{\prime} \leq 0$ for $n \leq x \leq n+2$. By the definition of solution of (3.1.1) we have that

$$
\begin{align*}
& \mid \int_{0}^{\infty} u(x, t) d x-\int_{0}^{\infty} u(x, 0) d x-\int_{0}^{t} \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right) u(x, s) d x d s \\
& \quad-\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \int_{0}^{x} \beta(x-y, y) u(x-y, s) u(y, s) d y d x d s \\
& \quad+\int_{0}^{t} \int_{0}^{\infty}\left[\int_{0}^{\infty} \beta(x, y) u(y, s) d y+m_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right)\right] u(x, s) d x d s \mid \\
= & \mid \int_{n}^{\infty}(u(x, t)-u(x, 0))(1-\zeta(x)) d x+\int_{0}^{t} \int_{n}^{\infty} g_{1}(x, s) \zeta_{x}(x) u(x, s) d x d s \\
& +\frac{1}{2} \int_{0}^{t} \int_{n}^{\infty}(\zeta(x)-1) \int_{0}^{x} \beta(x-y, y) u(x-y, s) u(y, s) d y d x d s \\
& +\int_{0}^{t} \int_{n}^{\infty} u(x, s)(1-\zeta(x))\left[\int_{0}^{\infty} \beta(x, y) u(y, s) d y+m_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right)\right] d x d s \mid \\
\leq & \left(2+T\left\|g_{1}\right\|_{\infty}+\frac{3}{2} T\|\beta\|_{\infty}\|u\|_{\mathbf{L}^{\infty}\left((0, T) ; \mathbf{L}^{1}(0, \infty)\right)}+T\left\|m_{1}\right\|_{\infty}\right) \sup _{t \in[0, T]} \int_{n}^{\infty} u(x, t) d x . \tag{3.4.12}
\end{align*}
$$

Since $u \in \mathbf{L}^{\infty}\left((0, T) ; \mathbf{L}^{1}(0, \infty)\right), \sup _{t \in[0, T]} \int_{n}^{\infty} u(x, t) d x \rightarrow 0$ as $n \rightarrow \infty$. Thus, (3.4.12) implies (3.4.11) holds.

To show $\varphi^{z}(t)$ are continuous on $[0, t]$, it suffices to show that

$$
\begin{align*}
\int_{0}^{\infty} z(x, t) d x= & \int_{0}^{\infty} z(x, 0) d x+\int_{0}^{t} \int_{0}^{\infty} \gamma_{2}\left(x, s, \varphi^{u}, \varphi^{z}\right) z(x, s) d x d s  \tag{3.4.13}\\
& -\int_{0}^{t} \int_{0}^{\infty} m_{2}\left(x, s, \varphi^{u}, \varphi^{z}\right) z(x, s) d x d s
\end{align*}
$$

Let $\eta(x, t)=\rho(x)$, where $\rho(x)=1$ for $0 \leq x \leq n, \rho(x)=0$ for $n+2 \leq x<\infty$ and
$-1 \leq \rho^{\prime}(x) \leq 0$ for $n \leq x \leq n+2$. By the definition of solution of (3.1.1) we have that

$$
\begin{align*}
& \left|\int_{0}^{\infty}(z(x, t)-z(x, 0)) d x-\int_{0}^{t} \int_{0}^{\infty}\left(\gamma_{2}\left(x, s, \varphi^{u}, \varphi^{z}\right)-m_{2}\left(x, s, \varphi^{u}, \varphi^{z}\right)\right) z(x, s) d x d s\right| \\
= & \mid \int_{n}^{\infty}(z(x, t)-z(x, 0))(1-\rho(x)) d x+\int_{0}^{t} \int_{n}^{\infty} g_{2}(x, s) \rho_{x}(x) z(x, s) d x d s \\
& +\int_{0}^{t} \int_{n}^{\infty} z(x, s)(1-\rho(x)) m_{2}\left(x, s, \varphi^{u}, \varphi^{z}\right) d x d s \mid \\
\leq & \left(2+T\left\|g_{2}\right\|_{\infty}+T\left\|m_{2}\right\|_{\infty}\right) \sup _{t \in[0, T]} \int_{n}^{\infty} z(x, t) d x . \tag{3.4.14}
\end{align*}
$$

Since $z \in \mathbf{L}^{\infty}\left((0, T) ; \mathbf{L}^{1}(0, \infty)\right), \sup _{t \in[0, T]} \int_{n}^{\infty} z(x, t) d x \rightarrow 0$ as $n \rightarrow \infty$. Thus, (3.4.14) implies (3.4.13) holds.

Theorem 3.4.4. Suppose that (H1)-(H5) hold. Then the unique solution (3.1.1) exists for $0 \leq t<\infty$.

Proof. By the definition of the solution of (3.1.1), we only need to show that $\varphi^{u}(t)$ and $\varphi^{z}(t)$ are global in $t$. By (3.4.11), we have that

$$
\begin{aligned}
\varphi^{u}(t)= & \varphi^{u}(0)+\int_{0}^{t} \int_{0}^{\infty} \gamma_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right) u(x, s) d x d s \\
& +\frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} \int_{0}^{x} \beta(x-y, y) u(x-y, s) u(y, s) d y d x d s \\
& -\int_{0}^{t} \int_{0}^{\infty}\left[\int_{0}^{\infty} \beta(x, y) u(y, s) d y+m_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right)\right] u(x, s) d x d s \\
= & \varphi^{u}(0)+\int_{0}^{t} \int_{0}^{\infty}\left(\gamma_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right)-m_{1}\left(x, s, \varphi^{u}, \varphi^{z}\right)-\frac{1}{2} \int_{0}^{\infty} \beta(x, y) u(y, s) d y\right) u(x, s) d x d s \\
\leq & \varphi^{u}(0)+\left\|\gamma_{1}\right\|_{\infty} \int_{0}^{t} \varphi^{u}(s) d s
\end{aligned}
$$

By Gronwall's inequality, we obtain that $\varphi^{u}(t) \leq \varphi^{u}(0) \exp \left(\left\|\gamma_{1}\right\|_{\infty} t\right)$. By (3.4.13), we have

$$
\begin{aligned}
\varphi^{z}(t) & =\varphi^{z}(0)+\int_{0}^{t} \int_{0}^{\infty}\left(\gamma_{2}\left(x, s, \varphi^{u}, \varphi^{z}\right)+m_{2}\left(x, s, \varphi^{u}, \varphi^{z}\right)\right) z(x, s) d x d s \\
& \leq \varphi^{z}(0)+\left\|\gamma_{2}\right\|_{\infty} \int_{0}^{t} \varphi^{z}(s) d s
\end{aligned}
$$

By Gronwall's inequality, we obtain that $\varphi^{z}(t) \leq \varphi^{z}(0) \exp \left(\left\|\gamma_{2}\right\|_{\infty} t\right)$.

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## Chapter 4

## Numerical Solver for General Size-Structured Population Model

### 4.1 Introduction

We wrote a user-friendly software package using Matlab and compiled them as standalone package to solve the following coupled system of size-structured population model:

$$
\left.\begin{array}{l}
u_{t}^{I}+\left(g^{I}(x,\right. \\
\left.\left.\quad P^{1}(x, t), P^{2}(x, t), \cdots, P^{N}(x, t)\right) u\right)_{x} \\
\quad+m^{I}\left(x, P^{1}(x, t), P^{2}(x, t), \cdots, P^{N}(x, t)\right) u=0, \quad(x, t) \in(0, L] \times(0, T] \\
g^{I}\left(0, P^{1}(0, t), P^{2}(0, t), \cdots, P^{N}(0, t)\right) u^{I}(0, t) \\
\quad=C^{I}(t)+\int_{0}^{L} \beta^{I}\left(x, P^{1}(x, t), P^{2}(x, t), \cdots, P^{N}(x, t)\right) u^{I}(x, t) d x, \quad t \in(0, T] \tag{4.1.1}
\end{array}\right\}
$$

Here $u^{I}(x, t), I=1,2, \ldots, N$ is the density of individuals of $I$ th subpopulation having size $x$ at time $t$, and

$$
\begin{equation*}
P^{I}(x, t)=\int_{0}^{L} w^{I}(x, y) u^{I}(y, t) d y \tag{4.1.2}
\end{equation*}
$$

is a function of density of $u^{I}$, where $N$ is the number of subpopulation, and $w^{I}(x, y)$ is a weighing function. The function $g^{I}$ denotes the growth rate of an individual in the $I$ th subpopulation, and $m^{I}$ denotes the mortality rate of an individual in the $I$ th subpopulation. The function $\beta^{I}$ is the reproduction rate of an individual in the $I$ th
subpopulation, while $C^{I}$ represents the inflow rate of zero-size individual from an external source. In fact, we can easily see that the model (4.1.1) is a generalization for models discussed in Chapter 1 and 2.

The remainder of this chapter is organized as follows. In section 4.2, we introduce some features about the main figure. In section 4.3, we introduce how to initialize a simulation. In section 4.4, we present some operations during a running simulation. In section 4.5, we introduce some features after simulation is completed. In the last section 4.6, we introduce how to create a stand alone package.

### 4.2 Main Figure

The main figure of the size-structured population program is where the majority of the interaction the user will used to create and run a simulation (see Figure 4.1). Two axes located in left of the figure are used to display the graph of Population Density and Total Population, respectively. By clicking on the Population popup menu, you can choose the subpopulation that you want to display its corresponding figures or its parameter values. The upper right hand of the figure is the status bar that gives the current status of the simulation. Below that is the number of populations, maximum time $(T)$, maximum size $(L)$, all the parameters value for a population and some calculation push button. More details are given below.

## - DEFAULT Button:

This initializes all the parameters to their default values.

## - STOP Button:

This stops a current simulation from finishing.

## - CONTINUE Button:

This continues a previously completed simulation.


Figure 4.1: Main figure of the numerical solver.

## - START Button:

This begins a new simulation.

- Number of Populations:

Number of subpopulation $(N)$. Only positive integers will be accepted. Otherwise, an error dialogue will be given. The default value is 1 .

- Maximum Time:

The maximum time of the simulation running $(T)$. Only positive values will be accepted. Otherwise, an error dialogue will be given. The default value is 1.5.

- Number of Time Steps:

The number of time steps of the numerical scheme. Only positive integer will be accepted. The time mesh size is then given by $T$ /number of time steps. The default value is 150 .

- Maximum Size:

The maximum size of the population $(L)$. Only positive values will be accepted. Otherwise, an error dialogue will be given. The default value is 1 .

- Number of Size Intervals:

The number of size intervals for the numerical scheme. Only positive integer will be accepted. Otherwise, an error dialogue will be given. The spacial mesh size is then given by $L /$ Number of Size Intervals. The default value is 100 .

- Initial Population Density, $u(x, 0)$ :
$u(x, 0)$ is a nonnegative function with respect to $x$. The default function is $3 \exp (-10(x-$ $\left.0.01)^{2}\right)$.
- Growth Rate, $g(x, P 1, P 2, \ldots, P N)$ : $g(x, P 1, P 2, \ldots, P N)$ is a nonnegative function depending on the variables $x, P 1$, $P 2, \ldots, P N$ if the model involves $N$ subpopulation. The default function is $5(1-$ $x) Q \exp (-2 Q)$, where $Q=P 1+P 2+\cdots+P N$.
- Mortality Rate, $m(x, P 1, P 2, \ldots, P N)$ :
$m(x, P 1, P 2, \ldots, P N)$ is a nonnegative function depending on the variables $x, P 1$, $P 2, \ldots, P N$ if the model involves $N$ subpopulation. The default function is $Q /(1+$ $Q)$, where $Q=P 1+P 2+\cdots+P N$.
- Reproduction Rate, $b(x, P 1, P 2, \ldots, P N)$ :
$b(x, P 1, P 2, \ldots, P N)$ is a nonnegative function depending on the variables $x, P 1$, $P 2, \ldots, P N$ if the model involves $N$ subpopulation. The default function is $0.2 x \exp (-0.2 Q)$, where $Q=P 1+P 2+\cdots+P N$.
- Inflow Rate of Zero-Size Individual, $C(t)$ :
$C(t)$ is a nonnegative function depending on the variable $t$. The default function is 0.
- Weighing Funtion, $w(x, y)$ :

If the problem is a hierarchical size-structured population model, then $w(x, y)$ is a nonnegative function depending on both variables $x$ and $y$. Otherwise, $w(x, y)$ is a nonnegative function only depending on the variable $y$. The default function is $(0.2(y \leq x)+(y>x)) y$.

### 4.3 Initializing A Simulation

This section explains how to initialize a simulation step by step in order to make it function properly.

1. Choose the number of subpopulations for the problems

Edit the number of subpopulations for the problems in the edit box under the message Number of Populations. The default value for the number of populations is 1 .
2. Edit parameter values using Default Setup or A Custom-Made Setup.

- Default Setup:
- Press the Default button. This creates a Quasilinear Hierarchical Size Structured Model in which the parameter values of Initial Population Density is assumed to be the same for all the subpopulations, so are Growth Rate, Mortality Rate, Reproductive Rate, Inflow of Zero-Size Individuals and Weighing Function.
- To see a subpopulation parameters, just click on the Population popup menu. If you want to change some of its parameters, just edit it in the corresponding edit box.
- A Custom-Made Simulation:
- Enter the maximum time $(T)$ and maximum size $(L)$ for your problem, and then enter the number of time steps and size intervals for the numerical scheme.
- Enter the appropriate function forms for all the parameters of Population 1. Inappropriate forms will lead to an error message (see Figure 4.2). If the number of populations for the problem is more than 1, repeat this process by first clicking on the Population popup menu for another successive subpopulation and then entering all of its parameters values.


Figure 4.2: Inappropriate function form result in error message.
3. Press the Start button to start this simulation.

### 4.4 Operations During A Running Simulation

This section explains what kind of operations that are available while a simulation is currently running.

- Counter:

Notice at the top right hand corner of the main figure, the status message is displaying a counter that increments by 1 . This is the number of time steps the model
has been running. The simulation will run until this counter reaches the Number of Time Steps, unless the user stops the program prematurely or the Continue button is pressed to start this simulation.

- Stopping a Simulation:

If you want to stop the simulation prematurely, just press the Stop button and the simulation will end.

### 4.5 Features After A Simulation Is Completed

This section explains what kind of operations can be taken place when a previous simulation is completed.

- Graphing:

You can see the graphs of Population Density and Total Population of a supopulation by clicking on the Population popup menu (see Figure 4.3). For example, if you want to see the graphs of population $2(N \geq 2)$, just drop down the Population popup menu, choose the menu Population 2.


Figure 4.3: Figure for simulation finished.

- Continuing a Simulation:

If you want to continue a previous prematurely completed simulation, then just press the Continue button. You can also change some the parameters value of a subpopulation (growth rate, reproduction rate, mortality rate and weighing function) by just editing them in the corresponding edit box. Press the Continue button to continue.

- Saving a Simulation:

This feature saves all of the parameters that the user enters and the all of the outputs such as the total population number, etc. To save your simulation, click on Save, which is located at the top of the main menu. The Saving Data window will appear (see Figure 4.4). On the right hand side is the current directory and on the left side is the files in the current directory. At the bottom right hand corner is where you can enter the name of the saved file. This file will be saved as a .mat file in the current directory. You can change the current directory by clicking on the current directory window. When you are ready to saved this file, just click on the OK button.


Figure 4.4: Figure for saving data.

- Loading a Simulation:

This feature loads all of the parameters that the user enters and the all of the outputs such as the total population, etc. To load a previous simulation, click on Load, which is located at the top left of the main figure. The Loading Data window will appear (see Figure 4.5). On the right hand side is the current directory and on the left side is the files in the current directory. At the bottom right hand corner is where you can enter the name of the load file or you can select the file from the file listing in the current directory. You can change the current directory by clicking on the current directory window. When you are ready to load this file, just click on the OK button.


Figure 4.5: Figure for loading previous data.

### 4.6 Creating A Stand-Alone Application

In this section, we will use an example to explain how to create a stand-alone graphics application in a step-by-step process. In order to do it, you must install MATLAB,

MATLAB Compiler, C++ Compiler, MATLAB C++ Math Library and MATLAB C++ Graphics Library in your system. Suppose that you have 3 MATLAB files: HierCommunity.m, Default.m and StartHierCommnuity.m, which are located in the directory C: $\backslash$ test. The process is as follows:

## - Configuring mbuild

To configure your compiler, run mbuild -setup from MATLAB command prompt. This allows you to choose the appropriate C compiler.

- Verifying mbuild

To verify that mbuild is properly configured on your system to create a standalone application, copy $<$ MATLAB $>\backslash$ extern $\backslash$ examples $\backslash$ cmath $\backslash$ ex1.c to your local directory and cd to that directory, where $<$ MATLAB $>$ represents your MATLAB installation directory. Then, at the MATLAB prompt, enter:

## mbuild ex1.c

This should create a file ex1.exe. To launch your application, enter its name on the command line. If mbuild is properly configured, you will get no error and obtain an answer.

- Verifying the MATLAB Compiler.

To verify the MATLAB Compiler, copy $<$ MATLAB $>\backslash$ extern $\backslash$ examples $\backslash$ compiler $\backslash$ hello.m to your local directory and cd to that directory, and then type the following at the MATLAB prompt:
mcc -em hello.m

This command should complete without errors.

- Create a new M-file as follows:
mcc -p -B sglcpp HierCommunity.m Default.m StartHierCommnuity.m

Then run it. You will see that a new subdirectory C: \test $\backslash$ bin is created. In the same time, all the M-files are translated into $\mathrm{C}++$ source code suitable for your own stand-alone external applications.

- Packaging the MATLAB Math Run-Time Libraries
- Run the MATLAB Math and Graphics Run-Time Library Installer by doubleclicking on the mglinstaller.exe file, which is located at $<$ MATLAB $>\backslash$ extern $\backslash$ lib \win32\mglinstaller.exe. This program extracts the libraries from the archive. You must install them in the directory $\mathrm{C}: \backslash$ test.
- You will see that another new subdirectory C: $\backslash$ test $\backslash$ toolbox is created. In the same time, all the dynamic link libraries are created in the new subdirectory $\mathrm{C}: \backslash$ test $\backslash$ bin $\backslash$ win32, copy all the files located in subdirectory win32 and then paste them in the directory $\mathrm{C}: \backslash$ test

Remark: There are restrictions on what kind of MATLAB code can be compiled. The MATLAB compiler cannot compile:

- Script M-files. If it is not a function file, just put a 'function *' line at the top, where ${ }^{*}$ denotes the file name.
- M-files containing inline or $\operatorname{eval}(\exp 1, \exp 2)$.
- M-file name is not the same as the name in the sentence 'function'.
- M-file containing audio command such as sound, etc.


## Conclusion

We used the finite difference approximation method to study the existence-uniqueness of the solution to a nonlinear hierarchical size-structured model in Chapter 1. The crucial step in this technique is to show that the developed finite difference approximation has a bounded total variation. Then, through the compact imbedding of the space of functions of bounded variation in $\mathcal{L}^{1}(0, L)$ one can extract a convergent subsequence and show that the limit is indeed a solution. This approach seems to be more applicable than that used in $[1,2]$. In particular, the approach used in [1] requires that vital rates depend only on the total population. The techniques used in [2] require that the growth and reproduction rates depend only on the size and the mortality rate depend only on the total population. However, for our approach, it is applicable to all these cases. Furthermore, application of such a technique results not only in the existence of solutions but also in a numerical scheme that can be used to investigate the solution quantitatively. Notice that the order of the convergence rate of the finite difference scheme developed in this chapter is only one, we hope to develop a numerical scheme whose convergence rate is second order for our model in the future work. Moreover, we want to consider other generalizations which will increase the applicability of our model. In particular, we wish to study the case where $Q(x, t)=\int_{0}^{L} d(x, y) u(y, t) d y$. Clearly this is a more general $Q$ than the one considered in our model. In fact, if $d(x, y)=\alpha w(y)$ for $0 \leq y \leq x$ and $d(x, y)=w(y)$ for $x<y \leq L$ then one obtains the environment $Q$ considered in this chapter. We believe that the finite difference approximation method used in this chapter works for this general $Q$. However,
additional technicality will be needed.
We developed a least square technique in Chapter 2 for parameter identification in a coupled system of nonlinear size-structured populations model. Recall that the observations in this chapter corresponds to the total population number instead of population density used in $[3,4]$. In practical situations it is impossible or there is much difficulties to obtain the data on the population density. So our method has more practical meaning than the one used in $[3,4]$. Furthermore, the numerical experiments show that this technique performs well and produce good confidence interval for the parameters. However, we see that there is a slightly under biased estimator for some of numerical examples when the infinite dimensional effects exist, we suspect that the upwind scheme we adopted to approximate the model and the right hand sum for approximating all the integrals may result in this bias. So we hope in the future to improve it by developing a new numerical scheme to approximating the infinite dimensional state and parameter space without losing the convergence, in the same time, we also hope this new numerical scheme can have a higher order convergence rate.

We used a monotone method in Chapter 3 to establish the existence-uniqueness for a nonlinear nonlocal size-structured phytoplankton-zooplankton aggregation model. The idea behind such a method is to replace the actual solution in all the nonlinear and nonlocal terms with some previous guess for the solution, then solve the resulting linear model to obtain a new guess for the solution. Iteration of such a procedure yields the solution of the original problem upon passing to the limit. The key step between the consecutive guesses is a comparison principle. Note that the growth rates in our model do not depend on the total population, we wish to develop a method in the future work that will deal with this case. We also want to take the dynamics of nutrient into account in the future.

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#### Abstract

In Chapter 1, a finite difference approximation to a hierarchical size-structured model with nonlinear growth, mortality and reproduction rates is developed. Existence-uniqueness of the weak solution to the model is established and convergence of the finite difference approximation is proved. Simulations indicate that the monotonicity assumption on the growth rate is crucial for the global existence of weak solutions. Numerical results testing the efficiency of this method in approximating the long-time behavior of the model are presented.

In Chapter 2, a least-squares technique is developed for identifying unknown parameters in a coupled system of nonlinear size-structured populations. Convergence results for the parameter estimation technique are established. Ample numerical simulations and statistical evidence are provided to demonstrate the feasibility of this approach.

In Chapter 3, we consider a nonlinear size-structured phytoplankton-zooplankton aggregation Model. We develop a comparison principle and construct monotone sequences to show the existence of the solution. The uniqueness of the solution is also established.


## BIOGRAPHICAL SKETCH

Shuhua Hu was born on November 12, 1976, in Weihai, Shandong Province, China. She attended Qingdao University in September 1994 and received a Bachelor of Science in Mathematics Education in July 1998. She was accepted for her graduate studies in September 1998 at Nanjing University of Aeronautics and Astronautics. After receiving her master's degree in Computational Mathematics in March 2001, she moved to the United States of America in August 2001 to work on her Ph.D. in Applied Mathematics at the University of Louisiana at Lafayette. She completed the degree requirements in December 2004.

